

RANKIN–EISENSTEIN CLASSES FOR MODULAR FORMS

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ABSTRACT. In this paper we make a systematic study of certain motivic cohomology classes (“Rankin–Eisenstein classes”) attached to the Rankin–Selberg convolution of two modular forms of weight ≥ 2 . The main result is the computation of the p -adic syntomic regulators of these classes. As a consequence we prove many cases of the Perrin-Riou conjecture for Rankin–Selberg convolutions of cusp forms.

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1. INTRODUCTION

1.1. **Background.** Rankin–Eisenstein classes were first introduced by Beilinson in [Bei84], in order to prove his general conjecture on the values of motivic L -functions for the value at $s = 2$ of the Rankin–Selberg convolution L -function $L(f, g, s)$ associated to two cusp forms of weight 2. The Rankin–Eisenstein classes of Beilinson live in the motivic cohomology group

$$H_{\text{mot}}^3(Y_1(N) \times Y_1(N), \mathbf{Q}(2))$$

where $Y_1(N)$ is the modular curve classifying elliptic curves with a point of exact order N , and are obtained by push-forward of modular units along the diagonal. Beilinson mainly considered the regulator into Deligne cohomology

$$r_{\mathcal{D}} : H_{\text{mot}}^3(Y_1(N) \times Y_1(N), \mathbf{Q}(2)) \rightarrow H_{\mathcal{D}}^3(Y_1(N)_{\mathbf{R}} \times Y_1(N)_{\mathbf{R}}, \mathbf{R}(2))$$

and was able to compute explicitly the image of Rankin–Eisenstein classes under $r_{\mathcal{D}}$. Beilinson’s work was generalized by Scholl to the case of higher weight cusp forms (unpublished, but see [Kin98] for similar results in the case of Hilbert modular surfaces).

Extending Beilinson’s ideas in a different direction, Flach studied the image under the étale regulator

$$r_{\text{ét}} : H_{\text{mot}}^3(Y_1(N) \times Y_1(N), \mathbf{Q}(2)) \rightarrow H_{\text{ét}}^3(Y_1(N) \times Y_1(N), \mathbf{Q}_p(2))$$

of certain motivic classes closely related to Rankin–Eisenstein classes, and used these to obtain finiteness results for the Tate–Shafarevich group of the symmetric square of an elliptic curve.

In recent years the program of Bertolini–Darmon–Rotger on the systematic study of Rankin–Selberg convolutions in p -adic families gave a new impulse to the study of Rankin–Eisenstein classes, which they call Beilinson–Flach elements. Roughly, the work of Bertolini–Darmon–Rotger is concerned with the syntomic regulator

$$r_{\text{syn}} : H_{\text{mot}}^3(Y_1(N) \times Y_1(N), \mathbf{Q}(2)) \rightarrow H_{\text{syn}}^3(Y_1(N)_{\mathbf{Z}_p} \times Y_1(N)_{\mathbf{Z}_p}, \mathbf{Q}_p(2))$$

and certain generalizations to p -adic families. One of the culmination points of their work are new results on the equivariant Birch–Swinnerton-Dyer conjecture for elliptic curves.

The authors’ research was supported by the following grants: SFB 1085 “Higher invariants” (Kings); Royal Society University Research Fellowship “ L -functions and Iwasawa theory” and NSF Grant No. 0932078 000 (Loeffler); EPSRC First Grant EP/J018716/1 and NSF Grant No. 0932078 000 (Zerbes).

The philosophy behind the conjectures of Bloch–Kato and Perrin-Riou made it plausible that one should be able to build an Euler system out of the Rankin–Eisenstein classes if one changes the level of $Y_1(N)$. This goal was achieved in the paper [LLZ14], which made it possible to use the powerful Euler system machinery also for Rankin–Selberg convolutions. However, the theory in [LLZ14] used only weight 2 Rankin–Eisenstein classes, and did not show that these were compatible in p -adic families with Rankin–Eisenstein classes of higher weight; and lacked strong results on the non-triviality of the Euler system, which are expected to result from an explicit reciprocity law.

1.2. Our results. In this paper and its sequel, we shall construct three-variable families interpolating the Rankin–Eisenstein classes of all possible weights, and prove an explicit reciprocity law relating these to L -functions. Our approach to the explicit reciprocity law is completely different from the ones used so far. The construction proceeds in two steps.

Firstly, we shall compute explicitly the syntomic regulators of the Rankin–Eisenstein classes (for cusp forms of level prime to p and arbitrary weights ≥ 2) and identify them with special values of a p -adic L -function. It is crucial to note that we are dealing here with non-critical values of the complex L -function, i.e. points outside the range of interpolation.

Secondly, we shall show that the étale classes vary in p -adic families as the weights change. The explicit reciprocity law is then a consequence of the compatibility between the étale and the syntomic regulator via the Bloch–Kato logarithm map, and the work of the second and third authors on p -adic interpolation of Bloch–Kato logarithms in families [LZ14].

As a consequence one gets an explicit relation between special values of the p -adic L -function at critical values, where it interpolates the complex L -function, with the image of the étale Rankin–Eisenstein classes under the dual exponential map. This is the desired explicit reciprocity law.

The idea of proving an explicit reciprocity law in this upside down way is inspired by the conjecture of Perrin-Riou on special values of p -adic L -functions, which is the p -adic analogue of the Beilinson conjecture. Her conjecture states roughly, that even at special values where the p -adic L -function does *not* interpolate the complex L -function, there is still a direct relation between the value of the complex L -function and the p -adic one.

That such a strategy for proving the explicit reciprocity law is successful relies on the following fact: The Rankin–Eisenstein classes for higher weight cusp forms are obtained by push-forward of Beilinson’s motivic Eisenstein classes via the diagonal to the product of modular curves

$$CG_{\text{mot}}^{[k,k',j]} : H_{\text{mot}}^i(Y, \text{TSym}^{k+k'-2j} \mathcal{H}_{\mathbf{Q}}(n)) \rightarrow H_{\text{mot}}^i(Y, \text{TSym}^k \mathcal{H}_{\mathbf{Q}} \otimes \text{TSym}^{k'} \mathcal{H}_{\mathbf{Q}}(n-j))$$

by what we call the Clebsch–Gordan map. But the syntomic and étale regulators of the Eisenstein classes were computed explicitly in [BK10] and [Kin13] already from the perspective of p -adic variation in families.

We would like to note that in [BDR14b] Bertolini, Darmon and Rotger have also proved an explicit reciprocity law for Rankin–Eisenstein classes (in a one-variable p -adic family). Their strategy is related to ours, but with the important difference that they do not consider classes associated to higher-weight modular forms. Rather, they interpolate from specializations of a Hida family which are classical modular forms of weight 2 and high p -power level, using the results of Besser and the second and third authors on syntomic regulators for varieties with bad reduction [BLZ15].

1.3. This paper. In this paper we will extend the computations mentioned above in the syntomic case to Rankin–Selberg convolutions obtaining a direct relation with Hida’s p -adic L -function. We will not touch more difficult questions of p -adic interpolation in the étale cohomology, which will be treated in the sequel [KLZ15]. Note that this paper and its sequel are both extensions and generalizations of our earlier preprint [KLZ14].

Let f, g be cuspidal new eigenforms of weights $k+2, k'+2 \geq 2$, levels N_f, N_g and characters $\varepsilon_f, \varepsilon_g$. Denote by $L(f, g, s)$ the L -function for the Rankin–Selberg convolution of f and g . We assume p is an odd prime that does not divide $N_f N_g$. Recall that Hida has constructed a p -adic L -function for with the following interpolation property: for s an integer in the range $k'+2 \leq s \leq k+1$, we have

$$L_p(f, g, s) = \frac{\mathcal{E}(f, g, s)}{\mathcal{E}(f)\mathcal{E}^*(f)} \cdot \frac{\Gamma(s)\Gamma(s-k'-1)}{\pi^{2s-k'-1}(-i)^{k-k'} 2^{2s+k-k'} \langle f, f \rangle_{N_f}} \cdot L(f, g, s),$$

where $\mathcal{E}(f, g, s)$, $\mathcal{E}(f)$ and $\mathcal{E}^*(f)$ are various Euler factors at p . Moreover, the function $L_p(f, g, s)$ varies analytically as f varies in a Hida family, and if g is ordinary it also varies analytically in g .

Now let $s = 1 + j$ where $0 \leq j \leq \min(k, k')$. Note that for these s we do not have a relation to the complex L -function. We shall define below a *syntomic Rankin–Eisenstein class*

$$\mathrm{Eis}_{\mathrm{syn},1,N}^{[k,k',j]} \in H_{\mathrm{syn}}^3(Y_1(N)^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)).$$

Projection to the (f, g) -isotypical component induces a map

$$\mathrm{AJ}_{\mathrm{syn},f,g} : H_{\mathrm{syn}}^3(Y_1(N)^2_{\mathbf{Z}_p}, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \longrightarrow t(M(f \otimes g)^*(-j))_{\mathbf{Q}_p},$$

which we call the syntomic Abel–Jacobi map, where

$$t(M(f \otimes g)^*(-j)) := M_{\mathrm{dR}}(f \otimes g)^* / \mathrm{Fil}^{-j} M_{\mathrm{dR}}(f \otimes g)^*.$$

is the tangent space of the dual of the motive $M(f \otimes g)$ associated to the Rankin–Selberg convolution of f and g . This tangent space is dual to $\mathrm{Fil}^{j+1} M_{\mathrm{dR}}(f \otimes g)$. Our first main result shows that the class

$$\mathrm{AJ}_{\mathrm{syn},f,g} \left(\mathrm{Eis}_{\mathrm{syn},1,N}^{[k,k',j]} \right) \in t(M(f \otimes g)^*(-j))_{\mathbf{Q}_p}$$

is directly related to the value of Hida’s p -adic L -function $L_p(f, g, s)$ at $s = j + 1$:

Theorem 1 (see Theorem 6.5.9). *Let $0 \leq j \leq \min(k, k')$. There is a certain differential form $\eta_f^\alpha \otimes \omega_g' \in \mathrm{Fil}^{j+1} M_{\mathrm{dR}}(f \otimes g)$ such that the natural pairing with $\mathrm{AJ}_{\mathrm{syn},f,g} \left(\mathrm{Eis}_{\mathrm{syn},1,N}^{[k,k',j]} \right)$ gives*

$$\left\langle \mathrm{AJ}_{\mathrm{syn},f,g} \left(\mathrm{Eis}_{\mathrm{syn},1,N}^{[k,k',j]} \right), \eta_f^\alpha \otimes \omega_g' \right\rangle = (-1)^{k'-j+1} (k')! \binom{k}{j} G(\varepsilon_f^{-1}) G(\varepsilon_g^{-1}) \frac{\mathcal{E}(f) \mathcal{E}^*(f)}{\mathcal{E}(f, g, 1+j)} L_p(f, g, 1+j),$$

where the factors $\mathcal{E}(f, g, 1+j)$, $\mathcal{E}(f)$ and $\mathcal{E}^*(f)$ are as above.

The special case $k = k' = j = 0$ of this theorem was proved in [BDR14a] (their result is formulated in terms of the value at $s = 2$, but one can use the functional equation to translate to $s = 1$; see [LLZ14, Theorem 5.6.4]). The above theorem extends this to all (k, k', j) at which the construction makes sense.

It is important to note that the complex L -function $L(f, g, s)$ vanishes to the first order at $s = j + 1$ so that there is no classical interpolation at this value. The idea of Perrin-Riou is that after inserting certain periods $\Omega_\infty(j + 1)$ and $\Omega_p(j + 1, v)$ one can extend the relation between the complex L -function and the p -adic one also to the points where there is no classical interpolation. To prove such a statement we make a computation in Deligne cohomology analogous to Theorem 1 in syntomic cohomology (replicating earlier unpublished work of Scholl). Combining this with Theorem 1, we obtain the main result of this paper, which is a special case of the conjecture of Perrin-Riou:

Theorem 2 (see Theorem 7.2.6). *Let $L_p(f, g, s)$ be Hida’s p -adic Rankin–Selberg L -function and let $0 \leq j \leq \min(k, k')$. Suppose that $\Omega_p(j + 1, v) \neq 0$ holds. Then*

$$\frac{L_p(f, g, j + 1)}{\Omega_p \left(j + 1, \frac{1-p^{-1}\varphi^{-1}}{1-\varphi} (-t)^{-j-1} v \right)} = 4^{-1} (-1)^{k'+1} \Gamma(j + 1) \Gamma(j - k')^* G(\varepsilon_f^{-1}) G(\varepsilon_g^{-1}) \frac{L'_{\{p\}}(f, g, j + 1)}{\Omega_\infty(j + 1)},$$

where $\Gamma(j - k')^* = \frac{(-1)^{k'-j}}{(k'-j)!}$ is the residue of $\Gamma(s)$ at $s = j - k'$.

Note that both sides of the above equation lie in L^\times . Perrin-Riou’s conjecture can also be formulated for $j < 0$, but in this range the order of vanishing of $L'(f, g, 1 + j)$ is 2, and our methods do not apply in this case.

We would like to mention that so far there are only very few cases where the conjecture of Perrin-Riou has been proven. Besides the case of Dirichlet L -functions (essentially treated by Perrin-Riou), only the cases of modular forms [Nik11] and certain elliptic curves with complex multiplication [BK11] are known.

1.4. Acknowledgements. The authors are very grateful to Massimo Bertolini, Henri Darmon, and Victor Rotger for many inspiring discussions, in which they shared with us their beautiful ideas about Beilinson–Flach elements. The authors also would like to thank the organizers of the Banff workshop “Applications of Iwasawa Algebras” in March 2013, at which the collaboration was initiated which led to this paper. Part of this paper was written while the second and third author were visiting MSRI in Autumn 2014; they would like to thank MSRI for the hospitality.

2. GEOMETRICAL PRELIMINARIES

In this section, we recall a number of cohomology theories attached to algebraic varieties (or more general schemes) over various base rings, and the relationships between these. The results of this section are mostly standard, but one aspect is new: we outline in §2.5 how to extend Besser’s finite-polynomial cohomology to cover general coefficient sheaves.

2.1. Cohomology theories. We begin by introducing several cohomology theories associated to algebraic varieties (or, more generally, schemes). The most fundamental of these is motivic cohomology:

Definition 2.1.1 (Beilinson, [Beĭ84]). *If X is a regular scheme, we define motivic cohomology groups*

$$H_{\text{mot}}^i(X, \mathbf{Q}(n)) := \text{Gr}_n^\gamma K_{2n-i}(X) \otimes \mathbf{Q},$$

the n -th graded piece of the γ -filtration of the $(2n - i)$ -th algebraic K -theory of X .

Remark 2.1.2. This definition is compatible with the definition due to Voevodsky used in [LLZ14]. Voevodsky’s motivic cohomology can be defined with \mathbf{Z} -coefficients, but in this paper we shall only consider motivic cohomology with \mathbf{Q} -coefficients (as we will need to decompose the motivic cohomology groups into eigenspaces for the action of a finite group), so the older definition via higher K -theory suffices.

We will also need to work with several other cohomology theories. In each of these there is an appropriate notion of a *coefficient sheaf*:

- étale cohomology, with coefficients in lisse étale \mathbf{Q}_p -sheaves (for schemes on which the prime p is invertible);
- algebraic de Rham cohomology (for smooth varieties over fields of characteristic 0), with coefficients in vector bundles equipped with a filtration and an integrable connection ∇ ;
- Betti cohomology (for smooth varieties over \mathbf{C}), with coefficients in \mathbf{Q} or more generally locally constant sheaves of \mathbf{Q} -vector spaces;
- Absolute Hodge cohomology (for smooth varieties over \mathbf{R} or \mathbf{C}), with coefficients in variations of mixed \mathbf{R} -Hodge structures;
- rigid cohomology for smooth \mathbf{Z}_p -schemes, with coefficients in overconvergent F -isocrystals;
- rigid syntomic cohomology for smooth \mathbf{Z}_p -schemes, with coefficients in the category of “admissible overconvergent filtered F -isocrystals” defined¹ in [Ban02, BK10].

We denote these theories by $H_{\mathcal{T}}^\bullet(\dots)$, for $\mathcal{T} \in \{\text{ét}, \text{dR}, B, \mathcal{H}, \text{rig}, \text{syn}\}$. We sometimes write $\overline{\text{ét}}$ for étale cohomology over $\overline{\mathbf{Q}}$. We write $\mathbf{Q}_{\mathcal{T}}$ for the trivial coefficient sheaf, and $\mathbf{Q}_{\mathcal{T}}(n)$ for the n -th power of the Tate object in the relevant category. (For Betti cohomology we take $\mathbf{Q}_B = \mathbf{Q}$, and $\mathbf{Q}_B(n) = (2\pi i)^n \mathbf{Q} \subseteq \mathbf{C}$.)

Remark 2.1.3. We shall occasionally abuse notation slightly by writing $H_{\mathcal{T}}^i(Y)$, for Y a scheme over a ring such as $\mathbf{Z}[1/N]$; in this case, we understand this to signify the cohomology of the base-extension of Y to a ring over which the cohomology theory \mathcal{T} makes sense. Thus, for instance, we write $H_{\text{syn}}^i(Y)$ or $H_{\mathcal{D}}^i(Y)$ for Y a scheme over $\mathbf{Z}[1/N]$, and by this we intend $H_{\text{syn}}^i(Y_{\mathbf{Z}_p})$ and $H_{\mathcal{D}}^i(Y_{\mathbf{R}})$ respectively. This convention makes it significantly easier to state theorems which hold in all of the above theories simultaneously.

For any of these theories, we can define higher direct images of coefficient sheaves for smooth proper morphisms, and we have a Leray spectral sequence. (For syntomic cohomology this was shown in the PhD thesis of N. Solomon, [Sol08].)

Remark 2.1.4. All our coefficients are of geometric origin, and in fact the cohomology with coefficients $H_{\mathcal{T}}^i(X, \mathcal{F}_{\mathcal{T}})$ we use arises as a direct summand of an appropriate $H_{\mathcal{T}}^i(Y, \mathbf{Q}_{\mathcal{T}})$ by decomposing the Leray spectral sequence for some map $Y \rightarrow X$.

2.2. Comparison maps. The above cohomology theories are related by a number of natural maps.

Regulator maps. Firstly, for each \mathcal{T} there is a “regulator” map

$$r_{\mathcal{T}} : H_{\text{mot}}^i(X, \mathbf{Q}(n)) \rightarrow H_{\mathcal{T}}^i(X, \mathbf{Q}_{\mathcal{T}}(n)).$$

These maps are compatible with cup-products, pullbacks, and pushforward along proper maps.

Remark 2.2.1. For the compatibility of the syntomic regulator r_{syn} with pushforward maps, see [DM12].

¹The definition of this coefficient category and the associated cohomology theory in fact depends not only on X , but also on a choice of a suitable smooth compactification \bar{X} ; but we shall generally suppress this from the notation.

Geometric comparison isomorphisms. We have the following well-known comparison isomorphisms. Firstly, if $X_{\mathbf{C}}$ is a smooth variety over \mathbf{C} one has a comparison isomorphism

$$(2.2.1) \quad H_{\mathrm{dR}}^i(X_{\mathbf{C}}, \mathbf{C}) \cong H_B^i(X_{\mathbf{C}}(\mathbf{C}), \mathbf{C}).$$

compatible with r_B, r_{dR} .

If $X_{\mathbf{C}} = X_{\mathbf{R}} \times_{\mathbf{R}} \mathbf{C}$ for some variety $X_{\mathbf{R}}$ over \mathbf{R} , then both sides have an \mathbf{R} -structure

$$H_{\mathrm{dR}}^i(X_{\mathbf{R}}, \mathbf{R}) \otimes \mathbf{C} \cong H_B^i(X_{\mathbf{R}}(\mathbf{C}), \mathbf{R}) \otimes \mathbf{C}$$

which are not respected by the comparison isomorphism, but $\mathrm{id} \otimes c$ on the left hand side (where $c : \mathbf{C} \rightarrow \mathbf{C}$ is complex conjugation) corresponds on $H_B^i(X_{\mathbf{R}}(\mathbf{C}), \mathbf{R}) \otimes \mathbf{C}$ to the map $\overline{F_{\infty}} = F_{\infty} \otimes c$, where F_{∞} denotes pullback via the complex conjugation automorphism F_{∞} of the topological space $X_{\mathbf{R}}(\mathbf{C})$. As usual we define $H_B^i(X_{\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n))^{\pm} := H_B^i(X_{\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n))^{\overline{F_{\infty}} = \pm 1}$ where $\mathbf{Q}(n) := (2\pi i)^n \mathbf{Q}$.

Similarly, let X be a smooth \mathbf{Z}_p -scheme, and suppose that we can embed X in a smooth proper \mathbf{Z}_p -scheme \bar{X} as the complement of a simple normal crossing divisor relative to $\mathrm{Spec} \mathbf{Z}_p$ (so that $\mathcal{X} = (X, \bar{X})$ is a *smooth pair* in the sense of [BK10, Appendix A]). Let \mathcal{F} be any admissible overconvergent filtered F -isocrystal on X ; then we can define de Rham and rigid realizations $\mathcal{F}_{\mathrm{dR}}$ and $\mathcal{F}_{\mathrm{rig}}$, and one has an isomorphism

$$(2.2.2) \quad H_{\mathrm{dR}}^i(X_{\mathbf{Q}_p}, \mathcal{F}_{\mathrm{dR}}) \cong H_{\mathrm{rig}}^i(X, \mathcal{F}_{\mathrm{rig}}).$$

If $X_{\mathbf{Q}_p}$ is a smooth variety over \mathbf{Q}_p one has Faltings comparison isomorphism

$$(2.2.3) \quad \mathrm{comp}_{\mathrm{dR}} : H_{\mathrm{dR}}^i(X_{\mathbf{Q}_p}, \mathbf{Q}_p) \otimes \mathbf{B}_{\mathrm{dR}} \cong H_{\mathrm{\acute{e}t}}^i(X_{\overline{\mathbf{Q}_p}}, \mathbf{Q}_p) \otimes \mathbf{B}_{\mathrm{dR}}$$

where \mathbf{B}_{dR} is Fontaine's ring of periods; and this isomorphism is $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ -equivariant, if we let the Galois group act trivially on $H_{\mathrm{dR}}^i(X_{\mathbf{Q}_p}/\mathbf{Q}_p)$ and via its native action on $H_{\mathrm{\acute{e}t}}^i(X_{\overline{\mathbf{Q}_p}}, \mathbf{Q}_p)$. This isomorphism is also compatible with the filtrations on both sides, and with the regulator maps $r_{\mathrm{\acute{e}t}}, r_{\mathrm{dR}}$ (for the variety $X_{\overline{\mathbf{Q}_p}}$).

If X is itself proper, then rigid cohomology coincides with crystalline cohomology, and one has a comparison isomorphism refining (2.2.3),

$$(2.2.4) \quad \mathrm{comp}_{\mathrm{cris}} : H_{\mathrm{rig}}^i(X, \mathbf{Q}_p) \otimes \mathbf{B}_{\mathrm{cris}} \cong H_{\mathrm{\acute{e}t}}^i(X_{\overline{\mathbf{Q}_p}}, \mathbf{Q}_p) \otimes \mathbf{B}_{\mathrm{cris}}$$

compatible with $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ and with the Frobenius φ .

Remark 2.2.2. One can check that if X is non-proper, but can be compactified to a smooth pair \mathcal{X} , then the étale cohomology of $X_{\mathbf{Q}_p}$ is crystalline. Hence, combining equations (2.2.2) and (2.2.3), we have a canonical isomorphism of the form (2.2.4); but it is not clear if it commutes with the action of φ when X is non-proper.

Remark 2.2.3. Note that we have not attempted to define a version of (2.2.3) or (2.2.4) with coefficients, as it is not clear what the appropriate category of coefficient sheaves should be.

2.3. The Leray spectral sequence and its consequences. For any of the cohomology theories $\mathcal{T} \in \{\acute{e}t, \bar{\acute{e}t}, \mathrm{dR}, B, \mathcal{H}, \mathrm{rig}, \mathrm{syn}\}$, and a variety X with structure map $\pi : X \rightarrow \mathrm{Spec} R_{\mathcal{T}}$ where $R_{\mathcal{T}}$ is the appropriate base ring, we have a Leray spectral sequence

$${}^{\mathcal{T}}E_2^{ij} = H_{\mathcal{T}}^i(\mathrm{Spec} R_{\mathcal{T}}, R^j \pi_* \mathcal{F}) \Rightarrow H_{\mathcal{T}}^{i+j}(X, \mathcal{F}).$$

For the “geometric” theories $\mathcal{T} \in \{\bar{\acute{e}t}, \mathrm{dR}, B, \mathrm{rig}\}$ this is not interesting, as the groups $H_{\mathcal{T}}^i(\mathrm{Spec} R_{\mathcal{T}}, -)$ are zero for $i \neq 0$. However, it is interesting for the “absolute” theories $\mathcal{T} \in \{\mathcal{D}, \acute{e}t, \mathrm{syn}\}$.

Deligne cohomology and absolute Hodge cohomology. For a smooth variety $a : X_{\mathbf{R}} \rightarrow \mathrm{Spec} \mathbf{R}$ one can define the Deligne-Beilinson cohomology groups $H_{\mathcal{D}}^i(X_{\mathbf{R}}, \mathbf{R}(n))$ in terms of holomorphic differentials with logarithmic poles along a compactification. These groups are connected with de Rham and Betti cohomology via a long exact cohomology sequence

$$(2.3.1) \quad \dots \rightarrow F^n H_{\mathrm{dR}}^i(X_{\mathbf{R}}, \mathbf{R}) \rightarrow H_B^i(X_{\mathbf{R}}(\mathbf{C}), \mathbf{R}(n-1))^+ \rightarrow H_{\mathcal{D}}^{i+1}(X_{\mathbf{R}}, \mathbf{R}(n)) \rightarrow F^n H_{\mathrm{dR}}^{i+1}(X_{\mathbf{R}}, \mathbf{R}) \rightarrow \dots$$

Recall the definition of absolute Hodge cohomology. Let $MHS_{\mathbf{R}}^+$ be the category of mixed \mathbf{R} -Hodge structures $M_{\mathbf{R}}$ carrying an involution $F_{\infty} : M_{\mathbf{R}} \rightarrow M_{\mathbf{R}}$, which respects the weight filtration and such that $\overline{F_{\infty}} : M_{\mathbf{R}} \otimes \mathbf{C} \rightarrow M_{\mathbf{R}} \otimes \mathbf{C}$ respects the Hodge filtration. For any separated scheme $X_{\mathbf{R}} \rightarrow \mathrm{Spec} \mathbf{R}$ of finite type Beilinson [Bei86] has defined a complex $R\Gamma(X_{\mathbf{R}}, \mathbf{R}(n)) \in D^b(MHS_{\mathbf{R}}^+)$ whose cohomology

groups are the mixed Hodge structures $H^i(X_{\mathbf{R}}(\mathbf{C}), \mathbf{R}(n))$ with the involution \overline{F}_∞ . The absolute Hodge cohomology of $X_{\mathbf{R}}$ is by definition

$$H_{\mathcal{H}}^i(X_{\mathbf{R}}, \mathbf{R}(n)) := R^i \operatorname{Hom}_{D^b(MHS_{\mathbf{R}}^+)}(\mathbf{R}(0), R\Gamma(X_{\mathbf{R}}, \mathbf{R}(n)))$$

and one has a short exact sequence

(2.3.2)

$$0 \rightarrow \operatorname{Ext}_{MHS_{\mathbf{R}}^+}^1(\mathbf{R}(0), H_B^{i-1}(X_{\mathbf{R}}(\mathbf{C}), \mathbf{R}(n))) \rightarrow H_{\mathcal{H}}^i(X_{\mathbf{R}}, \mathbf{R}(n)) \rightarrow \operatorname{Hom}_{MHS_{\mathbf{R}}^+}(\mathbf{R}(0), H_B^i(X_{\mathbf{R}}(\mathbf{C}), \mathbf{R}(n))) \rightarrow 0.$$

The computation of the Ext-groups of $M_{\mathbf{R}} \in MHS_{\mathbf{R}}^+$ is standard and one has

$$\operatorname{Hom}_{MHS_{\mathbf{R}}^+}(\mathbf{R}(0), M_{\mathbf{R}}) = W_0 M_{\mathbf{R}}^+ \cap \operatorname{Fil}^0 M_{\mathbf{C}}, \quad \operatorname{Ext}_{MHS_{\mathbf{R}}^+}^1(\mathbf{R}(0), M_{\mathbf{R}}) = \frac{W_0 M_{\mathbf{C}}^+}{W_0 M_{\mathbf{R}}^+ + \operatorname{Fil}^0 M_{\mathbf{C}}^+}.$$

In the case where $a : X_{\mathbf{R}} \rightarrow \operatorname{Spec} \mathbf{R}$ is smooth and the weights of $H_B^{i-1}(X_{\mathbf{R}}, \mathbf{R}(n))$ are ≤ 0 the absolute Hodge cohomology coincides with the Deligne-Beilinson cohomology $H_{\mathcal{D}}^i(X_{\mathbf{R}}, \mathbf{R}(n))$. The advantage of absolute Hodge cohomology is that one can define this theory also with coefficients.

Let $MHM_{\mathbf{R}}(X_{\mathbf{R}})$ be the category of algebraic \mathbf{R} -mixed Hodge modules over \mathbf{R} of Saito (this means a Hodge module over $X_{\mathbf{C}}$ together with an involution of \overline{F}_∞ , see [HW98, Appendix A] for more details). For any $M_{\mathbf{R}} \in MHM_{\mathbf{R}}(X_{\mathbf{R}})$ one defines

$$H_{\mathcal{H}}^i(X_{\mathbf{R}}, M_{\mathbf{R}}) := R^i \operatorname{Hom}_{MHM_{\mathbf{R}}(X_{\mathbf{R}})}(\mathbf{R}(0), M_{\mathbf{R}}).$$

In the case where $M_{\mathbf{R}} = \mathbf{R}(n)$ one has the adjunction

$$R \operatorname{Hom}_{MHM_{\mathbf{R}}(X_{\mathbf{R}})}(\mathbf{R}(0), \mathbf{R}(n)) \cong R \operatorname{Hom}_{D^b(MHS_{\mathbf{R}}^+)}(\mathbf{R}(0), Ra_* \mathbf{R}(n))$$

and $Ra_* \mathbf{R}(n) \cong R\Gamma(X_{\mathbf{R}}, \mathbf{R}(n))$. This interprets the above results in terms of the Leray spectral sequence for Ra_* .

Syntomic cohomology. The theory of syntomic cohomology, for smooth pairs (X, \bar{X}) over \mathbf{Z}_p , is closely parallel to that of absolute Hodge cohomology. An overconvergent filtered isocrystal on $\operatorname{Spec} \mathbf{Z}_p$ is simply a filtered φ -module, in the sense of p -adic Hodge theory, and the cohomology groups $H_{\operatorname{syn}}^i(\operatorname{Spec} \mathbf{Z}_p, D)$ are given by the cohomology of the 2-term complex $\operatorname{Fil}^0 D \xrightarrow{1-\varphi} D$; thus we have

$$(2.3.3) \quad H_{\operatorname{syn}}^0(\operatorname{Spec} \mathbf{Z}_p, D) = D^{\varphi=1} \cap \operatorname{Fil}^0 D, \quad H_{\operatorname{syn}}^1(\operatorname{Spec} \mathbf{Z}_p, D) = \frac{D}{(1-\varphi) \operatorname{Fil}^0 D}.$$

For a general smooth pair (X, \bar{X}) , and \mathcal{F} an admissible overconvergent filtered F-isocrystal on X , the comparison isomorphism $H_{\operatorname{dR}}^i(X_{\mathbf{Q}_p}, \mathcal{F}_{\operatorname{dR}}) \cong H_{\operatorname{rig}}^i(X, \mathcal{F}_{\operatorname{rig}})$ allows us to interpret these spaces as a filtered φ -module, which we shall denote by $H_{\operatorname{rig}}^i(X, \mathcal{F})$; this is precisely the higher direct image $R^i \pi_* \mathcal{F}$, where $\pi : X \rightarrow \operatorname{Spec} \mathbf{Z}_p$ is the structure map. Exactly as in the complex case, the Leray spectral sequence becomes a long exact sequence

$$(2.3.4) \quad \cdots \rightarrow H_{\operatorname{syn}}^i(X, \mathcal{F}) \rightarrow \operatorname{Fil}^0 H_{\operatorname{dR}}^i(X_{\mathbf{Q}_p}, \mathcal{F}_{\operatorname{dR}}) \xrightarrow{1-\varphi} H_{\operatorname{rig}}^i(X, \mathcal{F}_{\operatorname{rig}}) \rightarrow \cdots,$$

which is the p -adic analogue of the long exact sequence (2.3.1).

Étale cohomology. Let X be a smooth variety over a field K of characteristic 0, and \mathcal{F} a lisse étale sheaf on X . Then we have a Leray spectral sequence

$${}^{\operatorname{ét}} E_2^{ij} = H_{\operatorname{ét}}^i(\operatorname{Spec} K, H^j(X_{\bar{K}}, \mathcal{F})) \Rightarrow H_{\operatorname{ét}}^{i+j}(X, \mathcal{F}).$$

In general, this must be interpreted in terms of Jannsen's continuous étale cohomology [Jan88a]; we shall only use this for $K = \mathbf{Q}_p$, in which case continuous étale cohomology coincides with the usual étale cohomology.

If $X_{\mathbf{Q}}$ is a smooth variety over \mathbf{Q} , then $X_{\mathbf{Q}}$ admits a smooth model X over $\mathbf{Z}[1/S]$ for some set of primes S (with $p \in S$ without loss of generality); and if \mathcal{F} is of geometric origin, then it will extend to a lisse \mathbf{Q}_p -sheaf on X for large enough S . Then the cohomology groups $H^j(X_{\bar{\mathbf{Q}}}, \mathcal{F})$ are unramified outside S , and this sequence becomes

$$H^i(\operatorname{Gal}(\mathbf{Q}^S/\mathbf{Q}), H^j(X_{\bar{\mathbf{Q}}}, \mathcal{F})) \Rightarrow H_{\operatorname{ét}}^{i+j}(X, \mathcal{F}).$$

2.4. Compatibility of étale and syntomic cohomology. Let D be a filtered φ -module over \mathbf{Q}_p . As noted above, we can regard D as an overconvergent filtered F -isocrystal on $\mathrm{Spec} \mathbf{Z}_p$, and the cohomology groups $H_{\mathrm{syn}}^i(\mathrm{Spec} \mathbf{Z}_p, D)$ are given by the formulae (2.3.3).

If $D = \mathbf{D}_{\mathrm{cris}}(V)$ for V a crystalline $G_{\mathbf{Q}_p}$ -representation, then we have canonical maps

$$H_{\mathrm{syn}}^i(\mathrm{Spec} \mathbf{Z}_p, D) \rightarrow H^i(\mathbf{Q}_p, V)$$

arising from the Bloch–Kato short exact sequence of $G_{\mathbf{Q}_p}$ -modules

$$(2.4.1) \quad 0 \longrightarrow V \longrightarrow V \otimes \mathbf{B}_{\mathrm{cris}} \longrightarrow (V \otimes \mathbf{B}_{\mathrm{cris}}) \oplus (V \otimes \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+) \longrightarrow 0.$$

The map $H_{\mathrm{syn}}^i(\mathrm{Spec} \mathbf{Z}_p, D) \rightarrow H^i(\mathbf{Q}_p, V)$ is an isomorphism for $i = 0$, and for $i = 1$ it is injective, with image the subspace $H_f^1(\mathbf{Q}_p, V)$ parametrizing crystalline extensions of the trivial representation by V . The resulting isomorphism

$$\frac{D}{(1 - \varphi) \mathrm{Fil}^0 D} \xrightarrow{\cong} H_f^1(\mathbf{Q}_p, V)$$

is denoted by $\widetilde{\mathrm{exp}}_{\mathbf{Q}_p, V}$ in [LVZ14]; it satisfies $\widetilde{\mathrm{exp}}_{\mathbf{Q}_p, V} \circ (1 - \varphi) = \mathrm{exp}_{\mathbf{Q}_p, V}$, where $\mathrm{exp}_{\mathbf{Q}_p, V}$ is the Bloch–Kato exponential map.

Remark 2.4.1. If D is a filtered φ -module over \mathbf{Q}_p , then the space $D/(1 - \varphi) \mathrm{Fil}^0 D$ is easily seen to parametrize extensions (in the category of filtered φ -modules) of the trivial module by D , and the map $\widetilde{\mathrm{exp}}_{\mathbf{Q}_p, V}$ is just the natural map $\mathrm{Ext}_{\varphi, \mathrm{Fil}}^1(\mathbf{1}, \mathbf{D}_{\mathrm{cris}}(V)) \longrightarrow \mathrm{Ext}_{G_{\mathbf{Q}_p}}^1(\mathbf{1}, V)$.

Then we have the following theorem:

Theorem 2.4.2 (Besser, Niziol).

(1) Suppose X is smooth and quasi-projective. Then there is a natural map

$$\mathrm{comp} : H_{\mathrm{syn}}^i(X, \mathbf{Q}_p(n)) \rightarrow H_{\mathrm{ét}}^i(X_{\mathbf{Q}_p}, \mathbf{Q}_p(n)),$$

for each n , fitting into a commutative diagram

$$(2.4.2) \quad \begin{array}{ccc} H_{\mathrm{mot}}^i(X, \mathbf{Q}(n)) & \xrightarrow{r_{\mathrm{syn}}} & H_{\mathrm{syn}}^i(X, \mathbf{Q}_p(n)) \\ \downarrow & & \downarrow \mathrm{comp} \\ H_{\mathrm{mot}}^i(X_{\mathbf{Q}_p}, \mathbf{Q}(n)) & \xrightarrow{r_{\mathrm{ét}}} & H_{\mathrm{ét}}^i(X_{\mathbf{Q}_p}, \mathbf{Q}_p(n)). \end{array}$$

where the left vertical map is given by base extension.

(2) If X is projective, there is a morphism of spectral sequences ${}^{\mathrm{syn}}E^{ij} \rightarrow {}^{\mathrm{ét}}E^{ij}$ for each n , compatible with the morphism comp on the abutment; and on the E_2 page the morphisms

$$H_{\mathrm{syn}}^i(\mathrm{Spec} \mathbf{Z}_p, H_{\mathrm{rig}}^j(X, \mathbf{Q}_p(n))) \rightarrow H^i(\mathbf{Q}_p, H_{\mathrm{ét}}^j(X_{\overline{\mathbf{Q}_p}}, \mathbf{Q}_p(n)))$$

are given by the exact sequence (2.4.1) for $V = H_{\mathrm{ét}}^j(X_{\overline{\mathbf{Q}_p}}, \mathbf{Q}_p(n))$, together with the Faltings comparison isomorphisms $\mathrm{comp}_{\mathrm{dR}} : H_{\mathrm{rig}}^j(X, \mathbf{Q}_p) \cong H_{\mathrm{dR}}^j(X, \mathbf{Q}_p) \cong \mathbf{D}_{\mathrm{dR}}(H_{\mathrm{ét}}^j(X_{\overline{\mathbf{Q}_p}}, \mathbf{Q}_p))$ of (2.2.3).

Proof. See [Bes00b], Corollary 9.10 and Proposition 9.11. \square

We will use this in §5.4 below, to show that the Abel–Jacobi maps for syntomic and étale cohomology are related by the Bloch–Kato exponential map.

2.5. Finite-polynomial cohomology. In order to evaluate the syntomic regulators, we will make use of a family of cohomology theories defined by Besser [Bes00a], depending on a choice of polynomial $P \in 1 + T\mathbf{Q}_p[T]$; these reduce to syntomic cohomology when $P(T) = 1 - T$. Besser’s theory is described in *op.cit.* for coefficient sheaves of the form $\mathbf{Q}_p(n)$, and we briefly outline below how to extend this to more general coefficient sheaves.

Let X be a smooth \mathbf{Z}_p -scheme, and \mathcal{F} an admissible overconvergent filtered F -isocrystal on X (or, more precisely, on some smooth pair (X, \bar{X}) compactifying X).

Definition 2.5.1. For a polynomial $P \in 1 + T\mathbf{Q}_p[T]$, we define groups $H_{\mathrm{fp}}^i(X, \mathcal{F}, P)$ by replacing $1 - \varphi$ with $P(\varphi)$ in the definition of rigid syntomic cohomology with coefficients (cf. [BK10, Appendix A]).

We also define compactly-supported versions $H_{\mathrm{fp}, c}^i(X, \mathcal{F}, P)$ similarly (cf. [Bes12]).

Remark 2.5.2. When $\mathcal{F} = \mathbf{Q}_p(n)$ for some n , these groups reduce to Besser's original finite-polynomial cohomology (cf. [Bes00a]), but with a different numbering: in Besser's theory $H_{\text{syn}}^i(X, \mathbf{Q}_p(n))$ corresponds to taking $P(T) = 1 - T/p^n$, whereas if more general coefficients are allowed, it is more convenient to number in such a way that syntomic cohomology always corresponds to $P(T) = 1 - T$, whatever the value of n .

Exactly as in the case of syntomic cohomology (see (2.3.4) above), we have a long exact sequence

$$\cdots \rightarrow H_{\text{fp}}^i(X, \mathcal{F}, P) \rightarrow \text{Fil}^0 H_{\text{dR}}^i(X_{\mathbf{Q}_p}, \mathcal{F}_{\text{dR}}) \xrightarrow{P(\varphi)} H_{\text{rig}}^i(X, \mathcal{F}_{\text{rig}}, P) \rightarrow \cdots$$

and similarly for the compactly-supported variant; and there are “change-of- P ” maps fitting into a diagram

$$\begin{array}{ccccc} \cdots \rightarrow H_{\text{fp}}^i(X, \mathcal{F}, P) & \longrightarrow & \text{Fil}^0 H_{\text{dR}}^i(X_{\mathbf{Q}_p}, \mathcal{F}_{\text{dR}}) & \xrightarrow{P(\varphi)} & H_{\text{rig}}^i(X, \mathcal{F}_{\text{rig}}, P) \rightarrow \cdots \\ & \downarrow & \downarrow \text{id} & & \downarrow Q(\varphi) \\ \cdots \rightarrow H_{\text{fp}}^i(X, \mathcal{F}, PQ) & \longrightarrow & \text{Fil}^0 H_{\text{dR}}^i(X_{\mathbf{Q}_p}, \mathcal{F}_{\text{dR}}) & \xrightarrow{PQ(\varphi)} & H_{\text{rig}}^i(X, \mathcal{F}_{\text{rig}}, PQ) \rightarrow \cdots \end{array}$$

Definition 2.5.3. Exactly as in the case of Tate-twist coefficients in [Bes12, §2], we define cup products

$$H_{\text{fp}}^i(X, \mathcal{F}, P) \times H_{\text{fp},c}^j(X, \mathcal{G}, Q) \xrightarrow{\cup} H_{\text{fp},c}^{i+j}(X, \mathcal{F} \otimes \mathcal{G}, P \star Q),$$

where the polynomial $P \star Q$ is defined by the formula

$$\left(\prod_i (1 - \alpha_i T) \right) \star \left(\prod_j (1 - \beta_j T) \right) = \prod_{i,j} (1 - \alpha_i \beta_j T).$$

These cup-products are compatible with the change-of- P maps, in the obvious sense. They also satisfy a more subtle compatibility with the long exact sequence: the cup-product $H_{\text{fp}}^i(X, \mathcal{F}, P) \times H_{\text{fp},c}^j(X, \mathcal{G}, Q) \xrightarrow{\cup} H_{\text{fp},c}^{i+j}(X, \mathcal{F} \otimes \mathcal{G}, P \star Q)$ is compatible with the cup-products

$$H_{\text{fp}}^u(\text{Spec } \mathbf{Z}_p, H_{\text{rig}}^i(S, \mathcal{F}), P) \times H_{\text{fp}}^v(\text{Spec } \mathbf{Z}_p, H_{\text{rig},c}^j(S, \mathcal{G}), Q) \rightarrow H_{\text{fp}}^{u+v}(\text{Spec } \mathbf{Z}_p, H_{\text{rig},c}^{i+j}(S, \mathcal{F} \otimes \mathcal{G}), P \star Q).$$

If the polynomial P satisfies $P(p^{-1}) \neq 0$, and X is connected and has dimension d , then there is a canonical isomorphism

$$\text{tr}_{\text{fp},X} : H_{\text{fp},c}^{2d+1}(X, \mathbf{Q}_p(d+1), P) \xrightarrow{\cong} \mathbf{Q}_p$$

given by $\frac{1}{P(p^{-1})} \text{tr}_{\text{rig},X}$, where $\text{tr}_{\text{rig},X} : H_{\text{rig},c}^{2d}(X, \mathbf{Q}_p) \cong \mathbf{Q}_p$ is the trace map for rigid cohomology; the inclusion of the factor $\frac{1}{P(p^{-1})}$ makes this map compatible with the change-of- P maps. For polynomials P, Q with $(P \star Q)(p^{-1}) \neq 0$, we thus have a pairing

$$\langle -, - \rangle_{\text{fp},X} : H_{\text{fp}}^i(X, \mathcal{F}, P) \times H_{\text{fp},c}^{2d+1-i}(X, \mathcal{F}^\vee(d+1), Q) \rightarrow \mathbf{Q}_p$$

given by composing the cup-product with the map $\text{tr}_{\text{fp},X}$.

3. MODULAR CURVES AND MODULAR FORMS

We now introduce the specific geometric objects to which we will apply the general theory of the previous section: the modular curves $Y_1(N)$, and various coefficient sheaves on these curves.

3.1. Symmetric tensors. If H is an abelian group, we define the modules $\text{TSym}^k H$, $k \geq 0$, of symmetric tensors with values in H following [Kin13, §2.2]. By definition, $\text{TSym}^k H$ is the submodule of \mathfrak{S}_k -invariant elements in the k -fold tensor product $H \otimes \cdots \otimes H$ (while the more familiar $\text{Sym}^k H$ is the module of \mathfrak{S}_k -coinvariants).

The direct sum $\bigoplus_{k \geq 0} \text{TSym}^k H$ is equipped with a ring structure via symmetrization of the naive tensor product, so for $h \in H$ we have

$$(3.1.1) \quad h^{\otimes m} \cdot h^{\otimes n} = \frac{(m+n)!}{m!n!} h^{\otimes(m+n)}.$$

Remark 3.1.1. There is a natural algebra homomorphism $\mathrm{Sym}^\bullet H \rightarrow \mathrm{TSym}^\bullet H$, which becomes an isomorphism in degrees up to k after inverting $k!$. However, in the sequel to this paper we will be interested in the case where H is a \mathbf{Z}_p -module, and k varies in a p -adic family, so we cannot use this fact without losing control of the denominators involved; so we shall need to distinguish carefully between TSym and Sym .

Similarly, one can define $\mathrm{TSym}^k M$ for a module over any commutative ring A (e.g. for vector spaces over a field). Note that in general TSym^k does not commute with base change, and hence does not sheafify well. In the cases where we consider $\mathrm{TSym}^k(H)$, H is always a free module over the coefficient ring, so that this functor coincides with $\Gamma^k(H)$, the k -th divided power of H .

This functor sheafifies (on an arbitrary site), so that the above definitions and constructions carry over to sheaves of abelian groups. Thus, for any of the cohomology theories $\mathcal{T} \in \{B, \text{ét}, \mathrm{dR}, \mathrm{syn}, \mathrm{rig}, \mathcal{D}\}$, and \mathcal{F} an object of the appropriate category of coefficient sheaves, we can define objects $\mathrm{TSym}^k \mathcal{F}$. We use this to make the following key definition:

Definition 3.1.2. *Let $\pi : \mathcal{E} \rightarrow Y$ be an elliptic curve such that \mathcal{E} and Y are regular. For $\mathcal{T} \in \{B, \text{ét}, \mathrm{dR}, \mathrm{syn}, \mathrm{rig}, \mathcal{D}\}$, we define an element of the appropriate category of coefficient sheaves on Y by*

$$\mathcal{H}_{\mathcal{T}} = (R^1 \pi_* \mathbf{Q}_{\mathcal{T}})^\vee.$$

3.2. Lieberman's trick.

Definition 3.2.1. *For an integer $k \geq 0$, let \mathfrak{S}_k be the symmetric group on k letters, and let \mathfrak{T}_k be the semidirect product $\mu_2^k \rtimes \mathfrak{S}_k$. We define a character*

$$\begin{aligned} \varepsilon_k : \mathfrak{T}_k &\rightarrow \mu_2 \\ (\eta_1, \dots, \eta_k, \sigma) &\mapsto \eta_1 \cdots \eta_k \mathrm{sgn}(\sigma). \end{aligned}$$

(Cf. [Sch98, §A.1].)

Let $\pi : \mathcal{E} \rightarrow Y$ be an elliptic curve such that \mathcal{E} and Y are regular. Let $\pi^k : \mathcal{E}^k \rightarrow Y$ be the k -fold fibre product of \mathcal{E} over Y . On \mathcal{E} the group μ_2 acts via $[-1] : \mathcal{E} \rightarrow \mathcal{E}$, and on \mathcal{E}^k the symmetric group \mathfrak{S}_k acts by permuting the factors. This induces an action of the semi-direct product \mathfrak{T}_k on \mathcal{E}^k .

Definition 3.2.2. *Let*

$$H_{\mathrm{mot}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}}(j)) := H_{\mathrm{mot}}^{i+k}(\mathcal{E}^k, \mathbf{Q}(j+k))(\varepsilon_k)$$

be the ε_k -eigenspace.

The next result, which is a standard application of Lieberman's trick, justifies the above notation. Cf. [BK10, Lemma 1.5].

Theorem 3.2.3. *Let $\mathcal{T} \in \{B, \text{ét}, \bar{\text{ét}}, \mathrm{dR}, \mathrm{syn}, \mathrm{rig}, \mathcal{D}\}$, and $\mathbf{Q}_{\mathcal{T}}, \mathcal{H}_{\mathcal{T}}$ be the realizations of \mathbf{Q}, \mathcal{H} in the respective categories. Then one has isomorphisms*

$$H_{\mathcal{T}}^{i+k}(\mathcal{E}^k, \mathbf{Q}_{\mathcal{T}}(j+k))(\varepsilon_k) \cong H_{\mathcal{T}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathcal{T}}(j)).$$

Moreover, the regulator map $r_{\mathcal{T}}$ commutes with the action of \mathfrak{T}_k , and thus gives a map

$$r_{\mathcal{T}} : H_{\mathrm{mot}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}}(j)) \rightarrow H_{\mathcal{T}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathcal{T}}(j)).$$

Proof. This is immediate from the fact that $[-1]_*$ acts on $R^i \pi_* \mathbf{Q}_{\mathcal{T}}$ by multiplication with $[-1]^{2-i}$ and the isomorphism $\mathcal{H}_{\mathcal{T}} \cong R^1 \pi_* \mathbf{Q}_{\mathcal{T}}(1)$ induced from the Tate pairing once one has a Leray spectral sequence. \square

Remark 3.2.4. In the case when Y is a smooth \mathbf{Z}_p -scheme, we can use Lieberman's trick to extend the comparison morphisms comp and $\mathrm{comp}_{\mathrm{dR}}$, defined above for cohomology with coefficients that are twists of the Tate object, to sheaves of the form $\mathrm{TSym}^k \mathcal{H}$. By identifying $H_{\mathrm{syn}}^i(Y, \mathrm{TSym}^k(\mathcal{H})(r))$ with a direct summand of $H_{\mathrm{syn}}^{i+k}(\mathcal{E}^k, \mathbf{Q}_p(r+k))$, and similarly for étale and de Rham cohomology, we obtain maps

$$\mathrm{comp} : H_{\mathrm{syn}}^i(Y, \mathrm{TSym}^k(\mathcal{H}_{\mathbf{Q}_p})(r)) \rightarrow H_{\text{ét}}^i(Y_{\mathbf{Q}_p}, \mathrm{TSym}^k(\mathcal{H}_{\mathbf{Q}_p})(r))$$

and isomorphisms

$$\mathrm{comp}_{\mathrm{dR}} : H_{\mathrm{dR}}^i(Y_{\mathbf{Q}_p}, \mathrm{TSym}^k(\mathcal{H}_{\mathbf{Q}_p})(r)) \rightarrow \mathbf{D}_{\mathrm{cris}}(H_{\text{ét}}^i(Y_{\overline{\mathbf{Q}_p}}, \mathrm{TSym}^k(\mathcal{H}_{\mathbf{Q}_p})(r))).$$

3.3. Modular curves. We recall some notations for modular curves, following [Kat04, §§1–2]. For an integer $N \geq 5$, we shall write $Y_1(N)$ for the $\mathbf{Z}[1/N]$ -scheme representing the functor

$$S \mapsto \{\text{isomorphism classes } (E, P)\}$$

where S is a $\mathbf{Z}[1/N]$ -scheme, E/S is an elliptic curve, and $P \in E(S)$ is a section of exact order N (equivalently, an embedding of the constant group scheme $\mathbf{Z}/N\mathbf{Z}$ into E).

We use the same analytic uniformization of $Y_1(N)(\mathbf{C})$ as in [Kat04, 1.8]:

$$\begin{aligned} \Gamma_1(N) \backslash \mathbf{H} &\cong Y_1(N)(\mathbf{C}) \\ \tau &\mapsto (\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}), 1/N) \end{aligned}$$

where \mathbf{H} is the upper half plane and $\Gamma_1(N) := \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \} \subseteq \mathrm{SL}_2(\mathbf{Z})$.

Let $\mathrm{Tate}(q)$ be the Tate curve over $\mathbf{Z}((q))$, with its canonical differential ω_{can} . Let $\zeta_N := e^{2\pi i/N}$; then the pair $(\mathrm{Tate}(q), \zeta_N)$ defines a morphism $\mathrm{Spec} \mathbf{Z}[\zeta_N, 1/N][[q]] \rightarrow Y_1(N)$ and hence defines a cusp ∞ . The q -development at ∞ is compatible with the Fourier series in the analytic theory if one writes $q := e^{2\pi i\tau}$.

Remark 3.3.1. Note that the cusp ∞ is not defined over \mathbf{Q} in our model; so the q -expansions of elements of the coordinate ring of $Y_1(N)_{\mathbf{Q}}$, or more generally of algebraic differentials on $Y_1(N)$ with values in the sheaves $\mathrm{Sym}^k \mathcal{H}_{\mathbf{Q}}^{\vee}$, do not have q -expansion coefficients in \mathbf{Q} .

The curves $Y_1(N)$ are equipped with Hecke correspondences T_n and T'_n for each integer $n \geq 1$, defined as in [Kat04, §2.9 & §4.9].

Remark 3.3.2. Note that the action of the operators T_n on de Rham cohomology is given by the familiar q -expansion formulae, while the operators T'_n are the transposes of the T_n , which have no direct interpretation in terms of q -expansions when n is not coprime to the level.

3.4. Motives for Rankin convolutions. Let f, g be normalized cuspidal new eigenforms of weight $k+2, k'+2$, levels N_f, N_g and characters $\varepsilon(f), \varepsilon(g)$. We choose a number field L containing the coefficients of f and g .

From the work of Scholl [Sch90], one knows how to associate (Grothendieck) motives $M(f), M(g)$ with coefficients in L to f, g . We denote by $M(f \otimes g)$ the tensor product of these motives (over L), which is a 4-dimensional motive over \mathbf{Q} with coefficients in L .

Definition 3.4.1. For $\mathcal{T} \in \{\mathrm{dR}, \mathrm{rig}, B, \bar{\mathrm{et}}\}$ (the “geometric” cohomology theories) we write $M_{\mathcal{T}}(f \otimes g)$ for the \mathcal{T} -realization of M , which is the maximal $L \otimes_{\mathbf{Q}} \mathbf{Q}_{\mathcal{T}}$ -submodule of

$$H_{\mathcal{T}}^2(Y_1(N_f) \times Y_1(N_g), \mathrm{Sym}^{(k, k')} \mathcal{H}_{\mathcal{T}}^{\vee}) \otimes_{\mathbf{Q}} L$$

on which the Hecke operators $(T_{\ell}, 1)$ and $(1, T_{\ell})$ act as multiplication by the Fourier coefficients $a_{\ell}(f)$ and $a_{\ell}(g)$ respectively, for every prime ℓ (including $\ell \mid N_f N_g$).

Note that this direct summand lifts, canonically, to a direct summand of $H_{c, \mathcal{T}}^2$, since f and g are cuspidal.

The Hodge filtration of $M_{\mathrm{dR}}(f \otimes g)$ is given by

$$\dim_L \mathrm{Fil}^n M_{\mathrm{dR}}(f \otimes g) = \begin{cases} 4 & n \leq 0 \\ 3 & 0 < n \leq \min\{k, k'\} + 1 \\ 2 & \min\{k, k'\} + 1 < n \leq \max\{k, k'\} + 1 \\ 1 & \max\{k, k'\} + 1 < n \leq k + k' + 2 \\ 0 & k + k' + 2 < n. \end{cases}$$

In particular, if $n = 1 + j$ with $0 \leq j \leq \min(k, k')$, the space $\mathrm{Fil}^{1+j} M_{\mathrm{dR}}(f \otimes g) = \mathrm{Fil}^0 M_{\mathrm{dR}}(f \otimes g)(1 + j)$ has dimension 3 over L .

Definition 3.4.2. Dually, we write $M_{\mathcal{T}}(f \otimes g)^*$ for the maximal quotient of

$$H_{\mathcal{T}}^2(Y_1(N_f) \times Y_1(N_g), \mathrm{TSym}^{[k, k']} \mathcal{H}_{\mathcal{T}}(2)) \otimes_{\mathbf{Q}} L$$

on which the dual Hecke operators $(T'_{\ell}, 1)$ and $(1, T'_{\ell})$ act as multiplication by $a_{\ell}(f)$ and $a_{\ell}(g)$.

Remark 3.4.3. The twist by 2 implies that the Poincaré duality pairing

$$M_{\mathcal{T}}(f \otimes g) \times M_{\mathcal{T}}(f \otimes g)^* \rightarrow L \otimes_{\mathbf{Q}} \mathbf{Q}_{\mathcal{T}}$$

is well-defined and perfect, justifying the notation $M_{\mathcal{T}}(f \otimes g)^*$.

Definition 3.4.4. *Let*

$$t(M(f \otimes g)(j)) := \frac{M_{\text{dR}}(f \otimes g)(j)}{\text{Fil}^0 M_{\text{dR}}(f \otimes g)(j)} \quad t(M(f \otimes g)^*(-j)) := \frac{M_{\text{dR}}(f \otimes g)^*(-j)}{\text{Fil}^0 M_{\text{dR}}(f \otimes g)^*(-j)}$$

be the tangent spaces of the motives $M(f \otimes g)(j)$ and $M(f \otimes g)^(-j)$.*

Note that $t(M(f \otimes g)^*(-j))$ is the dual of the L -vector space $\text{Fil}^{1+j} M_{\text{dR}}(f \otimes g)$; so for $0 \leq j \leq \min(k, k')$ it is also 3-dimensional over L . This tangent space will be the target of the Abel–Jacobi maps in §5.4 below.

3.5. Rankin L -functions. Let f, g be cuspidal eigenforms of weights $r, r' \geq 1$, levels N_f, N_g and characters $\varepsilon_f, \varepsilon_g$. We define the Rankin L -function

$$L(f, g, s) = \frac{1}{L_{(N_f N_g)}(\varepsilon_f \varepsilon_g, 2s + 2 - r - r')} \sum_{n \geq 1} a_n(f) a_n(g) n^{-s},$$

where $L_{(N_f N_g)}(\varepsilon_f \varepsilon_g, s)$ denotes the Dirichlet L -function with the Euler factors at the primes dividing $N_f N_g$ removed.

Remark 3.5.1. This Dirichlet series differs by finitely many Euler factors from the L -function $L(\pi_f \otimes \pi_g, s)$ of the automorphic representation $\pi_f \otimes \pi_g$ of $\text{GL}_2 \times \text{GL}_2$ associated to f and g . In particular, it has meromorphic continuation to all of \mathbf{C} . It is holomorphic on \mathbf{C} unless $\langle \bar{f}, g \rangle \neq 0$, where $\bar{f} = \sum \bar{a}_n(f) q^n$, in which case it has a pole at $s = r$. If f and g are normalized newforms and $(N_f, N_g) = 1$, then $L(f, g, s) = L(\pi_f \otimes \pi_g, s)$.

Theorem 3.5.2 (Shimura, [Shi77]). *For integer values of s in the range $r' \leq s \leq r - 1$, we have*

$$\frac{L(f, g, s)}{\pi^{2s-r'+1} \langle f, f \rangle_{N_f}} \in \overline{\mathbf{Q}},$$

where $\langle f_1, f_2 \rangle_N$ is the Petersson scalar product of weight r modular forms defined by

$$\int_{\Gamma_1(N) \backslash \mathbf{H}} \overline{f_1(\tau)} f_2(\tau) \Im(\tau)^{r-2} dx \wedge dy.$$

More precisely, we have the Rankin–Selberg integral formula

$$(3.5.1) \quad L(f, g, s) = N^{r+r'-2s-2} \frac{\pi^{2s-r'+1} (-i)^{r-r'} 2^{2s+r-r'}}{\Gamma(s) \Gamma(s-r'+1)} \left\langle \bar{f}, g E_{1/N}^{(r-r')}(\tau, s-r+1) \right\rangle_N$$

where $N \geq 1$ is some integer divisible by N_f and N_g and with the same prime factors as $N_f N_g$, and $E_{1/N}^{(r-r')}(\tau, s-r+1)$ is a certain real-analytic Eisenstein series (cf. [LLZ14, Definition 4.2.1]), whose values for s in this range are nearly-holomorphic modular forms defined over $\overline{\mathbf{Q}}$.

The next theorem shows that the algebraic parts of the L -values $L(f, g, s)$, for s in the above range, can be p -adically interpolated.

Theorem 3.5.3 (Hida). *Let $p \nmid N_f N_g$ be a prime at which f is ordinary. Then there is a p -adic L -function $L_p(f, g) \in \overline{\mathbf{Q}}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma]]$ with the following interpolation property: for s an integer in the range $r' \leq s \leq r - 1$, we have*

$$L_p(f, g, s) = \frac{\mathcal{E}(f, g, s)}{\mathcal{E}(f) \mathcal{E}^*(f)} \cdot \frac{\Gamma(s) \Gamma(s-r'+1)}{\pi^{2s-r'+1} (-i)^{r-r'} 2^{2s+r-r'} \langle f, f \rangle_{N_f}} \cdot L(f, g, s),$$

where the Euler factors are defined by

$$\begin{aligned} \mathcal{E}(f) &= \left(1 - \frac{\beta_f}{p \alpha_f}\right), \\ \mathcal{E}^*(f) &= \left(1 - \frac{\beta_f}{\alpha_f}\right), \\ \mathcal{E}(f, g, s) &= \left(1 - \frac{p^{s-1}}{\alpha_f \alpha_g}\right) \left(1 - \frac{p^{s-1}}{\alpha_f \beta_g}\right) \left(1 - \frac{\beta_f \alpha_g}{p^s}\right) \left(1 - \frac{\beta_f \beta_g}{p^s}\right). \end{aligned}$$

Moreover, the function $L_p(f, g, s)$ varies analytically as f varies in a Hida family, and if g is ordinary it also varies analytically in g .

Remark 3.5.4.

- (1) The L -function $L_p(f, g, s)$ considered here is $N^{2s+2-r-r'} \mathcal{D}_p(f, g, 1/N, s)$ in the notation of [LLZ14, §5]. We include the power of N in the definition because it makes $L_p(f, g, s)$ independent of the choice of N .
- (2) The complex L -function $L(f, g, s)$ is symmetric in f and g , i.e. we have $L(f, g, s) = L(g, f, s)$, but this is not true of $L_p(f, g, s)$.
- (3) The construction of $L_p(f, g, s)$ has recently been extended to the non-ordinary case by Urban [Urb14], who has constructed a three-parameter p -adic L -function with f, g varying over the Coleman–Mazur eigencurve; but we shall only consider the case of ordinary f, g in this paper.

4. EISENSTEIN CLASSES ON $Y_1(N)$

4.1. Motivic Eisenstein classes. The fundamental input to the constructions of this paper are the following cohomology classes first constructed by Beilinson:

Theorem 4.1.1. *Let $N \geq 5$ and let $b \in \mathbf{Z}/N\mathbf{Z}$ be nonzero. Then there exist nonzero cohomology classes (“motivic Eisenstein classes”)*

$$\mathrm{Eis}_{\mathrm{mot}, b, N}^k \in H_{\mathrm{mot}}^1(Y_1(N), \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}}(1))$$

for all integers $k \geq 0$, satisfying the following residue formula: we have

$$\mathrm{res}_{\infty} \left(\mathrm{Eis}_{\mathrm{mot}, b, N}^k \right) = -N^k \zeta(-1-k).$$

Proof. By the construction in [BL94, §6.4] there is associated to the canonical order N section t_N of the universal elliptic curve \mathcal{E} over $Y_1(N)$ a class $\mathcal{E}_{\mathrm{mot}, b}^{k+2} \in H_{\mathrm{mot}}^1(Y_1(N), \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}}(1))$, which is essentially the specialization of the elliptic polylogarithm at bt_N . It has the property that [BL94, 6.4.5]

$$r_{\mathrm{ét}}(\mathcal{E}_{\mathrm{mot}, b}^{k+2}) = -N^{k-1} \mathrm{contr}_{\mathcal{H}_{\mathbf{Q}_p}}((bt_N)^* \mathrm{pol}^{k+1})$$

where the right hand side is the notation of [Kin13, §4.2]. Now we set $\mathrm{Eis}_{\mathrm{mot}, b, N}^k := -N \mathcal{E}_{\mathrm{mot}, b}^{k+2}$; by the residue formula of [Kin13, Theorem 5.2.2] we see that $\mathrm{Eis}_{\mathrm{mot}, b, N}^k$ has the stated residue at ∞ . \square

Remark 4.1.2. Note that the residue formulae of [Kin13] include an extra factor of N , since the residue map at ∞ of $Y_1(N)$ and $Y(N)$ differ by this factor; and a factor of $k!$ arising from the fact that the canonical map $\mathbf{Z}_p \cong \mathrm{Sym}^k \mathbf{Z}_p \rightarrow \mathrm{TSym}^k \mathbf{Z}_p \cong \mathbf{Z}_p$ is multiplication by $k!$ (cf. [Kin13, Lemma 5.1.6]).

Remark 4.1.3. For $k = 0$, we have $H_{\mathrm{mot}}^1(Y_1(N), \mathbf{Q}(1)) = \mathcal{O}(Y_1(N))^{\times} \otimes \mathbf{Q}$, and the Eisenstein class $\mathrm{Eis}_{\mathrm{mot}, b, N}^k$ is simply the Siegel unit $g_{0, b/N}$ in the notation of [Kat04].

4.2. Eisenstein classes in other cohomology theories. As a consequence of the existence of the motivic Eisenstein class, we immediately obtain Eisenstein classes in all the other cohomology theories introduced above: we define

$$\mathrm{Eis}_{\mathcal{T}, b, N}^k = r_{\mathcal{T}} \left(\mathrm{Eis}_{\mathrm{mot}, b, N}^k \right).$$

In particular, we have the following:

- An étale Eisenstein class

$$\mathrm{Eis}_{\mathrm{ét}, b, N}^k \in H_{\mathrm{ét}}^1(Y_1(N)[1/p], \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1)).$$

- A de Rham Eisenstein class

$$\mathrm{Eis}_{\mathrm{dR}, b, N}^k \in H_{\mathrm{dR}}^1(Y_1(N)_{\mathbf{Q}}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}}(1)).$$

- An Eisenstein class in absolute Hodge cohomology

$$\mathrm{Eis}_{\mathcal{H}, b, N}^k \in H_{\mathcal{H}}^1(Y_1(N)_{\mathbf{R}}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{R}}(1))$$

whose image in $H_{\mathrm{dR}}^1(Y_1(N)_{\mathbf{R}}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{R}}(1))$ coincides with the image of $\mathrm{Eis}_{\mathrm{dR}, b, N}^k$ under $\otimes \mathbf{R}$.

- A syntomic Eisenstein class

$$\mathrm{Eis}_{\mathrm{syn}, b, N}^k \in H_{\mathrm{syn}}^1(\mathcal{Y}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1)),$$

for any $p \nmid N$, where \mathcal{Y} denotes the smooth pair $(Y_1(N)_{\mathbf{Z}_p}, X_1(N)_{\mathbf{Z}_p})$, whose image in the group $H_{\mathrm{dR}}^1(Y_1(N)_{\mathbf{Q}_p}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1))$ coincides with the image of $\mathrm{Eis}_{\mathrm{dR}, b, N}^k$ under $\otimes \mathbf{Q}_p$, and whose image under the map

$$\mathrm{comp} : H_{\mathrm{syn}}^1(\mathcal{Y}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1)) \rightarrow H_{\mathrm{ét}}^1(Y_1(N)_{\mathbf{Q}_p}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1))$$

is the localization at p of the étale Eisenstein class $\mathrm{Eis}_{\mathrm{ét}, b, N}^k$.

We shall give explicit formulae for the de Rham, absolute Hodge, and syntomic Eisenstein classes in §§4.3–4.5 below; these formulae involve Eisenstein series – classical algebraic Eisenstein series for the de Rham case, and real-analytic and p-adic Eisenstein series in the absolute Hodge and syntomic cases respectively.

4.3. The de Rham Eisenstein class. We give an explicit formula for the de Rham Eisenstein class $\text{Eis}_{\text{dR},b,N}^k$, in terms of certain modular forms which are Eisenstein series of weight $k+2$.

Definition 4.3.1. Write $\zeta_N := e^{2\pi i/N}$ and $q := e^{2\pi i\tau}$. For $k \geq -1$ and $b \in \mathbf{Z}/N\mathbf{Z}$ not equal to zero, we define an algebraic Eisenstein series

$$F_{k+2,b} := \zeta(-1-k) + \sum_{n>0} q^n \sum_{\substack{dd'=n \\ d,d'>0}} d^{k+1} (\zeta_N^{bd'} + (-1)^k \zeta_N^{-bd'})$$

where $\zeta(s) := \sum_{n>0} \frac{1}{n^s}$ is the Riemann zeta function.

Remark 4.3.2. The Eisenstein series $F_{k+2,b}$ is $F_{0,b/N}^{(k+2)}$ in the notation of [Kat04, Proposition 3.10] and $\frac{2}{N^{k+1}} G_{k+2}$ in the notation of [Kat76, Equation (2.4.5)].

As in Section 3.3, denote by $\text{Tate}(q)$ the Tate elliptic curve over $\mathbf{Z}((q))$. Endowing $\text{Tate}(q)$ with the order N section corresponding to $\zeta_N \in \mathbf{G}_m/q^{\mathbf{Z}}$ gives a point of $Y_1(N)$ over $\mathbf{Z}((q)) \otimes_{\mathbf{Z}} \mathbf{Z}[1/N, \zeta_N]$. Moreover, the canonical differential ω_{can} on $\text{Tate}(q)$ gives a section trivializing $\text{Fil}^1 \mathcal{H}^{\vee}|_{\text{Tate}(q)}$, and hence of $\text{Fil}^0 \mathcal{H}|_{\text{Tate}(q)}$, via the isomorphism $\mathcal{H} \cong \mathcal{H}^{\vee}(1)$; we write $v^{[0,k]}$ for the k -th tensor power of this section, regarded as a generator of $\text{TSym}^k \mathcal{H}$.

Proposition 4.3.3. The pullback of the de Rham Eisenstein class to $(\text{Tate}(q), \zeta_N)$ is given by

$$\text{Eis}_{\text{dR},b,N}^k = -N^k F_{k+2,b} \cdot v^{[0,k]} \otimes \frac{dq}{q}.$$

Proof. See [BK10]. (Our normalizations are slightly different from those of *op.cit.*, but it is clear that the above class has the correct residue at ∞ .) \square

4.4. The Eisenstein class in absolute Hodge cohomology. The results in this section are well-known and due to Beilinson and Deninger. As we have to express the computations with our normalizations anyway, we decided to show how the Eisenstein class in absolute Hodge cohomology can be computed very easily as in the syntomic case by solving explicitly a differential equation. This transfers the main idea in [BK10] from syntomic cohomology to the case of absolute Hodge cohomology. The aim of this section is to give an explicit description of the class $\text{Eis}_{\mathcal{H},b,N}^k := r_{\mathcal{H}}(\text{Eis}_{\text{mot},b,N}^k)$.

Proposition 4.4.1. The group $H_{\mathcal{H}}^1(Y_1(N)_{\mathbf{R}}, \text{TSym}^k \mathcal{H}_{\mathbf{R}}(1))$ is the group of equivalence classes of pairs $(\alpha_{\infty}, \alpha_{\text{dR}})$, where

$$\alpha_{\infty} \in \Gamma(Y_1(N)(\mathbf{C}), \text{TSym}^k \mathcal{H}_{\mathbf{R}} \otimes \mathcal{C}^{\infty})$$

is a \mathcal{C}^{∞} -section of $\text{TSym}^k(\mathcal{H}_{\mathbf{R}})(1)$ and $\alpha_{\text{dR}} \in \Gamma(X_1(N)_{\mathbf{R}}, \text{TSym}^k \underline{\omega} \otimes \Omega_{X_1(N)}^1(C))$ is an algebraic section with logarithmic poles along $C := X_1(N) \setminus Y_1(N)$, such that

$$\nabla(\alpha_{\infty}) = \pi_1(\alpha_{\text{dR}}).$$

A pair $(\alpha_{\infty}, \alpha_{\text{dR}})$ is equivalent to 0 if we have

$$(\alpha_{\infty}, \alpha_{\text{dR}}) = (\pi_1(\beta), \nabla(\beta)) \text{ for some } \beta \in \Gamma(X_1(N)_{\mathbf{R}}, \text{TSym}^k(\underline{\omega})(C)).$$

Here $\pi_1 : \text{TSym}^k \mathcal{H}_{\mathbf{C}} \cong \text{TSym}^k \mathcal{H}_{\mathbf{R}} \otimes \mathbf{C} \rightarrow \text{TSym}^k \mathcal{H}_{\mathbf{R}}(1)$ is induced by the projection $\mathbf{C} \rightarrow \mathbf{R}(1)$, $z \mapsto (z - \bar{z})/2$.

Proof. Can be deduced either from the explicit description as group of extensions of mixed Hodge modules or from the standard description of $H_{\mathcal{D}}^{k+1}(\mathcal{E}_{\mathbf{R}}^k, \mathbf{R}(k+1))$ and application of the projector ε_k . \square

Consider the covering $\mathbf{H} \rightarrow Y_1(N)(\mathbf{C})$, which maps $\tau \mapsto (\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}), 1/N)$. Over \mathbf{H} we have the standard section $\omega = dz$ of $\mathcal{H}_{\mathbf{C}}^{\vee}$, where z is the coordinate on \mathbf{C} . Denote by $\langle \ , \ \rangle$ the Poincaré duality

pairing on $\mathcal{H}_{\mathbf{C}}^{\vee}$ and by $\omega^{\vee}, \bar{\omega}^{\vee}$ the basis dual to $\omega, \bar{\omega}$. We have the following formulae:

$$\begin{aligned} \langle \bar{\omega}, \omega \rangle &= \frac{\tau - \bar{\tau}}{2\pi i} & \langle \omega, \bar{\omega} \rangle &= -\langle \bar{\omega}, \omega \rangle \\ \omega^{\vee} &= \frac{2\pi i \bar{\omega}}{\tau - \bar{\tau}} & \bar{\omega}^{\vee} &= -\frac{2\pi i \omega}{\tau - \bar{\tau}} \\ \nabla(\omega) &= \frac{\omega - \bar{\omega}}{\tau - \bar{\tau}} d\tau & \nabla(\bar{\omega}) &= \frac{\omega - \bar{\omega}}{\tau - \bar{\tau}} d\bar{\tau} \\ \nabla(\omega^{\vee}) &= -\frac{\omega^{\vee} d\tau + \bar{\omega}^{\vee} d\bar{\tau}}{\tau - \bar{\tau}} & \nabla(\bar{\omega}^{\vee}) &= \frac{\omega^{\vee} d\tau + \bar{\omega}^{\vee} d\bar{\tau}}{\tau - \bar{\tau}} \end{aligned}$$

Definition 4.4.2. Let $w^{(r,s)} := \omega^r \bar{\omega}^s \in \text{Sym}^k \mathcal{H}_{\mathbf{C}}^{\vee}$ and $w^{[r,s]} := \omega^{\vee, [r]} \bar{\omega}^{\vee, [s]} \in \text{TSym}^k \mathcal{H}_{\mathbf{C}}$.

Under the isomorphism $\mathcal{H}_{\mathbf{C}}^{\vee} \cong \mathcal{H}_{\mathbf{C}}$ induced by the pairing $\langle \cdot, \cdot \rangle$ the image of $w^{(j,k-j)}$ is given by

$$(-1)^j j! (k-j)! \left(\frac{\tau - \bar{\tau}}{2\pi i} \right)^k w^{[k-j, j]}.$$

One has $\overline{w^{(r,s)}} = w^{(s,r)}$ and $\overline{w^{[r,s]}} = (-1)^{r+s} w^{[s,r]}$.

We have

$$\nabla(w^{[r,s]}) = \frac{1}{\tau - \bar{\tau}} \left((-rd\tau + sd\bar{\tau}) w^{[r,s]} + (r+1) w^{[r+1, s-1]} - (s+1) w^{[r-1, s+1]} \right),$$

and hence, if D_j are C^{∞} functions of τ satisfying the symmetry relation

$$(2\pi i)^{k-j} (\tau - \bar{\tau})^{k-j} D_{k-j} = \overline{(2\pi i)^j (\tau - \bar{\tau})^j D_j},$$

we have

$$\begin{aligned} (4.4.1) \quad \nabla \left(\sum_{j=0}^k (2\pi i)^j (\tau - \bar{\tau})^j D_j w^{[k-j, j]} \right) \\ = \sum_{j=0}^k \left[(2\pi i)^{j+1} (\tau - \bar{\tau})^j (\delta_{2j-k}(D_j) + (k-j) D_{j+1}) d\tau + (\dots) d\bar{\tau} \right] w^{[k-j, j]}, \end{aligned}$$

where \dots indicate the term derived from the previous one by interchanging j and $k-j$ and applying complex conjugation. Here δ_r denotes the Maass–Shimura differential operator $(2\pi i)^{-1} \left(\frac{\partial}{\partial \tau} + \frac{r}{\tau - \bar{\tau}} \right)$.

We define real analytic Eisenstein series:

Definition 4.4.3. For $t \geq 0$ and $s \in \mathbf{C}$ with $t + 2\Re(s) > 2$ and $b \in \mathbf{Z}/N\mathbf{Z}$, we define

$$F_{t,s,b}^{\text{an}} := (-1)^t \frac{\Gamma(s+t)}{(2\pi i)^{s+t}} \sum_{(m,n) \in \mathbf{Z}^2 \setminus \{(0,0)\}} \frac{e^{2\pi i m b/N} (\tau - \bar{\tau})^s}{(m\tau + n)^t |m\tau + n|^{2s}}.$$

Remark 4.4.4. In [LLZ14, 4.2.1] this Eisenstein series was denoted by $F_{b/N}^{(t)}(\tau, s)$. Note also that $F_{k+2,0,b}^{\text{an}}$ coincides with the algebraic Eisenstein series $F_{k+2,b}$ of Definition 4.3.1.

With these definitions the Eisenstein class in absolute Hodge cohomology is given as follows:

Proposition 4.4.5. The class $\text{Eis}_{\mathcal{H},b,N}^k = (\alpha_{\infty}, \alpha_{\text{dR}}) \in H_{\mathcal{H}}^1(Y_1(N)_{\mathbf{R}}, \text{TSym}^k \mathcal{H}_{\mathbf{R}}(1))$ is given by

$$\alpha_{\infty} := \frac{-N^k}{2} \sum_{j=0}^k (-1)^j (k-j)! (2\pi i)^{j-k} (\tau - \bar{\tau})^j F_{2j-k, k+1-j, b}^{\text{an}} w^{[k-j, j]}$$

and

$$\alpha_{\text{dR}} := N^k F_{k+2,0,b}^{\text{an}} (-2\pi i) (\tau - \bar{\tau})^k w^{[0,k]} d\tau.$$

Proof. Note that $(\alpha_{\infty}, \alpha_{\text{dR}})$ does define a class in absolute Hodge cohomology, since the Eisenstein series $F_{t,s,b}$ satisfy $\delta_t F_{t,s,b} = F_{t+2, s-1, b}$ (cf. [LLZ14, Proposition 4.2.2(iii)]), and hence all terms of the sum in Equation (4.4.1) are zero except for $j = 0$ and $j = k$.

For $k > 0$, the Eisenstein class $\text{Eis}_{\mathcal{H},b,N}^k$ is uniquely determined by $\alpha_{\text{dR}} = \text{Eis}_{\text{dR},b,N}^k$; and α_{dR} as defined above satisfies this, since $F_{k+2,0,b}^{\text{an}}$ coincides with the holomorphic Eisenstein series $F_{k+2,b}$ of the previous section.

For $k = 0$ the Eisenstein class is characterized by $\text{Eis}_{\mathcal{H},b,N}^0 = (\log |g_{0,b/N}|, d \log(g_{0,b/N}))$, and we have

$$F_{0,1,b}^{\text{an}} = -2 \log |g_{0,b/N}|. \quad \square$$

Remark 4.4.6. Note that there is a typographical error in formula (iii) of [Kat04, (3.8.4)] (a minus sign is missing), which we incautiously reproduced without checking in [LLZ14, Proposition 4.2.2(v)].

4.5. The syntomic Eisenstein class on the ordinary locus. We review the description from [BK10] of the syntomic Eisenstein class $\text{Eis}_{\text{syn},b,N}^k \in H_{\text{syn}}^1(\mathcal{Y}^{\text{ord}}, \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1))$ in terms of p -adic Eisenstein series. We assume here that $p \nmid N$.

In this section we let Y^{ord} be the open subscheme of $Y = Y_1(N)_{\mathbf{Z}_p}$ where the Eisenstein series $E_{p-1} \in \Gamma(Y, \underline{\omega}^{\otimes p-1})$ is invertible. Let \mathcal{Y}, \mathcal{X} be the formal completions with respect to the special fibre and $\mathcal{Y}_{\mathbf{Q}_p}, \mathcal{X}_{\mathbf{Q}_p}$ be the associated rigid analytic spaces. We also let $j : \mathcal{Y}_{\mathbf{Q}_p} \rightarrow \mathcal{X}_{\mathbf{Q}_p}$ be the open immersion and $Y_{\mathbf{Q}_p}^{\text{an}}$ be the rigid analytic space associated to $Y_{\mathbf{Q}_p}$ so that $Y_{\mathbf{Q}_p}^{\text{an}}$ is a strict neighbourhood of j . Let \mathcal{Y} and \mathcal{Y}^{ord} be the smooth pairs (Y, X) and (Y^{ord}, X) . We shall give an explicit formula for the image of $\text{Eis}_{\text{syn},b,N}^k$ under the restriction map

$$H_{\text{syn}}^1(\mathcal{Y}, \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1)) \rightarrow H_{\text{syn}}^1(\mathcal{Y}^{\text{ord}}, \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1)).$$

These syntomic cohomology groups have an explicit presentation exactly parallel to Proposition 4.4.1 above:

Proposition 4.5.1 ([BK10, Proposition A.16]). *A class in $H_{\text{syn}}^1(\mathcal{Y}^{\text{ord}}, \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1))$ is given by a pair of sections $(\alpha_{\text{rig}}, \alpha_{\text{dR}})$, where*

$$\alpha_{\text{rig}} \in \Gamma(\mathcal{X}_{\mathbf{Q}_p}, j^{\dagger} \text{TSym}^k \mathcal{H} |_{Y_{\mathbf{Q}_p}^{\text{an}}})$$

is an overconvergent section,

$$\alpha_{\text{dR}} \in \Gamma(X_{\mathbf{Q}_p}, \text{TSym}^k \underline{\omega} \otimes \Omega_{X_{\mathbf{Q}_p}}^1(\text{Cusp}))$$

is an algebraic section with logarithmic poles along $\text{Cusp} := X_{\mathbf{Q}_p} \setminus Y_{\mathbf{Q}_p}$, and we have the relation

$$\nabla(\alpha_{\text{rig}}) = (1 - \varphi)\alpha_{\text{dR}}.$$

The natural map $H_{\text{syn}}^1(\mathcal{Y}, \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1)) \rightarrow H_{\text{dR}}^1(Y_{\mathbf{Q}_p}, \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1))$ is given by mapping $(\alpha_{\text{rig}}, \alpha_{\text{dR}})$ to the class of α_{dR} ; thus α_{dR} must be the unique algebraic differential with logarithmic poles at the cusps representing $\text{Eis}_{\text{dR},b,N}^k$, whose q -expansion was given in Proposition 4.3.3 above.

Before we can write down the explicit formula for α_{rig} , we need to introduce certain p -adic Eisenstein series, and a certain trivialization of $\mathcal{H}_{\mathbf{Q}_p}$.

Definition 4.5.2. Write $\zeta_N := e^{2\pi i/N}$ and $q := e^{2\pi i\tau}$. For $t, s \in \mathbf{Z}$ with $t + s \geq 1$ we set

$$F_{t,s,b}^{(p)} := \sum_{n>0} q^n \sum_{\substack{dd'=n \\ p \nmid d' \\ d,d'>0}} d^{t+s-1} (d')^{-s} (\zeta_N^{bd'} + (-1)^t \zeta_N^{-bd'}) \in \mathbf{Z}_p[\zeta_N][[q]].$$

Remark 4.5.3. For (t, s) satisfying the inequalities $s + t \geq 1$, $t \geq 1$, $s \leq 0$, the real-analytic Eisenstein series $F_{t,s,b}^{\text{an}}$ is an algebraic nearly-holomorphic modular form defined over \mathbf{Q} , and the p -adic one $F_{t,s,b}^{(p)}$ is the p -stabilization of this form; cf. §5.2 of [LLZ14].

Let $\tilde{\mathcal{Y}}$ be the formal scheme which classifies elliptic curves over p -adic rings with a $\Gamma_1(N)$ -structure together with an isomorphism $\eta : \hat{\mathbb{G}}_m \cong \hat{\mathcal{E}}$ of formal groups. The elements in $\Gamma(\tilde{\mathcal{Y}}, \mathcal{O}_{\tilde{\mathcal{Y}}})$ are called Katz p -adic modular forms. The discussion in [BK10, §5.2] shows that $F_{t,s,b}^{(p)}$ is the q -expansion of a p -adic modular form.

Denote by $\tilde{\mathcal{H}}_{\mathbf{Q}_p}$ the pull-back of $\mathcal{H}_{\mathbf{Q}_p}$ to $\tilde{\mathcal{Y}}$. Its dual $\tilde{\mathcal{H}}_{\mathbf{Q}_p}^{\vee}$ contains a canonical section $\tilde{\omega}$ with $\eta^*(\tilde{\omega}) = dT/(1+T)$.

Definition 4.5.4. Denote by $\xi \in \Omega_{\tilde{\mathcal{Y}}/\mathbf{Z}_p}^1$ the differential form which corresponds to $\tilde{\omega}^{\otimes 2}$ under the Kodaira-Spencer isomorphism $\underline{\omega}^{\otimes 2} \cong \Omega_{\tilde{\mathcal{Y}}/\mathbf{Z}_p}^1$ and denote by θ the differential operator dual to ξ .

Then define $\tilde{u} := \nabla(\theta)(\tilde{\omega}) \in \tilde{\mathcal{H}}_{\mathbf{Q}_p}^{\vee}$. The element \tilde{u} is a generator of the unit root space. We denote by $\tilde{\omega}^{\vee}, \tilde{u}^{\vee}$ the basis dual to $\tilde{\omega}, \tilde{u}$.

The actions of ∇ and φ on these vectors are given by the formulae

$$\begin{aligned}\nabla(\tilde{\omega}) &= \tilde{u} \otimes \xi, & \nabla(\tilde{\omega}^\vee) &= 0, \\ \nabla(\tilde{u}) &= 0, & \nabla(\tilde{u}^\vee) &= \tilde{\omega}^\vee \otimes \xi, \\ \varphi(\tilde{\omega}) &= p\tilde{\omega}, & \varphi(\tilde{\omega}^\vee) &= p^{-1}\tilde{\omega}^\vee, \\ \varphi(\tilde{u}) &= \tilde{u}, & \varphi(\tilde{u}^\vee) &= \tilde{u}^\vee.\end{aligned}$$

We are interested in the sheaves $\mathrm{Sym}^k \mathcal{H}^\vee$ and $\mathrm{TSym}^k \mathcal{H}$. The pullbacks of these to $\tilde{\mathcal{Y}}$ have bases of sections given, respectively, by the sections $v^{(r,s)} := \tilde{\omega}^r \tilde{u}^s$ with $r+s = k$, and by the $v^{[r,s]} := (\tilde{\omega}^\vee)^{[r]} (\tilde{u}^\vee)^{[s]}$ with $r+s = k$. The pairing is given by $\langle v^{[r,s]}, v^{(r',s')} \rangle = \delta_{rr'} \delta_{ss'}$, so these two bases are dual to each other.

We have

$$\begin{aligned}\nabla(v^{(r,s)}) &= r v^{(r-1,s+1)} \otimes \xi, & \nabla(v^{[r,s]}) &= (r+1) v^{[r+1,s-1]} \otimes \xi, \\ \varphi(v^{(r,s)}) &= p^r v^{(r,s)}, & \varphi(v^{[r,s]}) &= p^{-r} v^{[r,s]}.\end{aligned}$$

Remark 4.5.5. The Tate curve $\mathrm{Tate}(q)$, equipped with its natural isomorphism of formal groups to \mathbb{G}_m and the point ζ_N of order N , defines a point of $\tilde{\mathcal{Y}}$ over the ring $\mathbf{Z}_p[\zeta_N][[q]]$; the pullback of ξ is $\frac{dq}{q}$ and θ acts as the differential operator dual to this, which is $q \frac{d}{dq}$. This identifies our sections $\tilde{\omega}$ and \tilde{u} with the ω_{can} and η_{can} of [DR14, Eq. (22)], and θ with the Serre differential d of §2.4 of *op.cit.*

Since θ acts on q -expansions as $q \frac{d}{dq}$, we have the formula

$$\theta(F_{t,s,b}^{(p)}) = F_{t+2,s-1,b}^{(p)}$$

for any t, s with $t+s \geq 1$.

Definition 4.5.6. *Let*

$$\alpha_{\mathrm{rig}} := -N^k \sum_{j=0}^k (-1)^{k-j} (k-j)! F_{2j-k,k+1-j,b}^{(p)} v^{[k-j,j]} \in \Gamma(\tilde{\mathcal{Y}}, \mathrm{TSym}^k \tilde{\mathcal{H}}_{\mathbf{Q}_p})$$

and (as above) let

$$\alpha_{\mathrm{dR}} := -N^k F_{k+2,b} v^{[0,k]} \otimes \xi \in \Gamma(X_{\mathbf{Q}_p}, \mathrm{TSym}^k \underline{\omega} \otimes \Omega_{X_{\mathbf{Q}_p}}^1(\mathrm{Cusp}))$$

be the section representing $\mathrm{Eis}_{\mathrm{dR},b,N}^k$.

It is shown in [BK10, Lemma 5.10] that $\alpha_{\mathrm{rig}} \in \Gamma(\mathcal{X}_{\mathbf{Q}_p}, j^* \mathrm{TSym}^k \mathcal{H} |_{Y_{\mathbf{Q}_p}^{\mathrm{an}}})$. With these notations we have the following explicit description of the class $\mathrm{Eis}_{\mathrm{syn},b,N}^k$; note its similarity to the description of $\mathrm{Eis}_{\mathcal{H},b,N}^k$ in Proposition 4.4.5:

Theorem 4.5.7 ([BK10, Theorem 5.11]). *The class $\mathrm{Eis}_{\mathrm{syn},b,N}^k \in H_{\mathrm{syn}}^1(\mathcal{Y}^{\mathrm{ord}}, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(1))$ is given by $(\alpha_{\mathrm{rig}}, \alpha_{\mathrm{dR}})$ defined above.*

Proof. An immediate calculation gives

$$\nabla(\alpha_{\mathrm{rig}}) = -N^k F_{k+2,0,b}^{(p)} v^{[0,k]} \otimes \xi,$$

and we calculate that

$$(1-\varphi)(\alpha_{\mathrm{dR}}) = -N^k F_{k+2,0,b}^{(p)} v^{[0,k]} \otimes \xi,$$

Thus the pair $(\alpha_{\mathrm{rig}}, \alpha_{\mathrm{dR}})$ does define a class in syntomic cohomology, which maps to $\mathrm{Eis}_{\mathrm{dR},b,N}^k$ in de Rham cohomology.

For $k > 0$ this is enough to uniquely characterize the syntomic Eisenstein class, as the map from syntomic to de Rham cohomology is injective in this case (see [BK10, Proposition 4.1]). For $k = 0$ this does not hold; but the motivic Eisenstein class $\mathrm{Eis}_{\mathrm{mot},b,N}^0$ is just the Siegel unit $g_{0,b/N}$, and one knows that

$$r_{\mathrm{syn}}(g_{0,b/N}) = ((1-\varphi) \log g_{0,b/N}, \mathrm{dlog} g_{0,b/N}).$$

An easy series calculation shows that we have

$$\alpha_{\mathrm{rig}} = -F_{0,1,b}^{(p)} = (1-\varphi) \log g_{0,b/N}$$

as rigid-analytic sections of the sheaf $\mathbf{Q}_p(1)$ on Y^{ord} , as required. \square

5. RANKIN–EISENSTEIN CLASSES ON PRODUCTS OF MODULAR CURVES

We shall now define “Rankin–Eisenstein classes” as the pushforward of Eisenstein classes along maps arising from the diagonal inclusion $Y_1(N) \hookrightarrow Y_1(N)^2$.

5.1. The Clebsch–Gordan map. In this section, we’ll establish some results on tensor products of the modules $\mathrm{TSym}^k H$. Let H be any abelian group.

Let k, k', j be integers satisfying the inequalities

$$(5.1.1) \quad k \geq 0, \quad k' \geq 0, \quad 0 \leq j \leq \min(k, k').$$

By definition, we have

$$\mathrm{TSym}^{k+k'-2j} H \subseteq \mathrm{TSym}^{k-j} H \otimes \mathrm{TSym}^{k'-j} H.$$

On the other hand, the map

$$\wedge^2 H \rightarrow H \otimes H,$$

given by mapping $x \wedge y$ to the antisymmetric tensor $x \otimes y - y \otimes x$, gives a map

$$\mathrm{TSym}^j (\wedge^2 H) \rightarrow \mathrm{TSym}^j H \otimes \mathrm{TSym}^j H$$

by raising to the j -th power. Taking the tensor product of these two maps and using the multiplication in the tensor algebra $\mathrm{TSym}^\bullet H$, we obtain a map

$$CG^{[k,k',j]} : \mathrm{TSym}^{k+k'-2j}(H) \otimes \mathrm{TSym}^j(\wedge^2 H) \rightarrow \mathrm{TSym}^k(H) \otimes \mathrm{TSym}^{k'}(H).$$

We are interested in the case where $H \cong \mathbf{Z}^2$, in which case $\wedge^2 H = \det(H)$.

Definition 5.1.1. *Define*

$$CG^{[k,k',j]} : \mathrm{TSym}^{k+k'-2j}(H) \rightarrow \mathrm{TSym}^k(H) \otimes \mathrm{TSym}^{k'}(H) \otimes \det(H)^{-j}$$

to be the map defined by the above construction.

We will need the following explicit formula for a piece of the Clebsch–Gordan map. Composing the Clebsch–Gordan map $CG^{[k,k',j]}$ with the natural contraction map

$$\left(\mathrm{Sym}^k H^\vee \right) \otimes \left(\mathrm{Sym}^{k'} H^\vee \right) \otimes \left(\mathrm{TSym}^k(H) \otimes \mathrm{TSym}^{k'}(H) \right) \rightarrow \mathbf{Z}$$

gives a trilinear form

$$(5.1.2) \quad \left(\mathrm{Sym}^k H^\vee \right) \otimes \left(\mathrm{Sym}^{k'} H^\vee \right) \otimes \left(\mathrm{TSym}^{k+k'-2j} H \right) \rightarrow \det(H)^{-j}.$$

Let us fix a basis u, v of H and write $w^{[r,s]} = u^{[r]}v^{[s]} \in \mathrm{TSym}^{r+s} H$. We let u^\vee, v^\vee be the dual basis of H^\vee , and write $w^{(r,s)} = (u^\vee)^r (v^\vee)^s \in \mathrm{Sym}^{r+s} H^\vee$, so the bases $\{w^{[r,s]} : r+s=k\}$ of $\mathrm{TSym}^k H$ and $\{w^{(r,s)} : r+s=k\}$ of $\mathrm{Sym}^k H^\vee$ are dual to each other. We let e_1 be the basis $u \wedge v$ of $\det H$, and $e_j = e_1^{\otimes j}$.

Proposition 5.1.2. *The trilinear form (5.1.2) sends the basis vector*

$$w^{(0,k)} \otimes w^{(k',0)} \otimes w^{[s,t]}$$

to zero unless $(s,t) = (k'-j, k-j)$, in which case it is mapped to

$$\frac{k!(k')!}{j!(k-j)!(k'-j)!} \otimes e_{-j}.$$

Proof. An unpleasant computation shows that for $0 \leq s \leq k+k'-2j$, the Clebsch–Gordan map sends the basis vector $w^{[s,k+k'-2j-s]}$ of $\mathrm{TSym}^{k+k'-2j} H$ to the element

$$\sum_{r+r'=s} \sum_{i=0}^j (-1)^i \frac{(r+i)!(k-r+i)!(r'+j-i)!(k'-r'+j-i)!}{r!(r')!(k-r-j)!(k'-r'-j)!i!(j-i)!} w^{[r+i,k-r-i]} \otimes w^{[r'+j-i,k'-j+i-r']} \otimes e_{-j}$$

of $\mathrm{TSym}^k H \otimes \mathrm{TSym}^{k'} H \otimes \det(H)^{-j}$. The vector $w^{[r+i,k-r-i]} \otimes w^{[r'+j-i,k'-j+i-r']}$ pairs nontrivially with $w^{(0,k)} \otimes w^{(k',0)}$ if and only if $r=i=0$ and $r'=s=k'-j$, and substituting these values gives the formula claimed. \square

5.2. Geometric realization of the Clebsch–Gordan map. The constructions of the previous section can clearly also be carried out with sheaves of abelian groups; so for any of our cohomology theories $\mathcal{T} \in \{B, \mathrm{dR}, \acute{\mathrm{e}}\mathrm{t}, \tilde{\acute{\mathrm{e}}}\mathrm{t}, \mathrm{rig}, \mathrm{syn}, \mathcal{H}\}$ for which we have a well-behaved category of coefficients, and $E \rightarrow Y$ an elliptic curve with E and Y regular, we obtain Clebsch–Gordan maps

$$CG_{\mathcal{T}}^{[k,k',j]} : \mathrm{TSym}^{k+k'-2j} \mathcal{H}_{\mathcal{T}} \rightarrow \mathrm{TSym}^k \mathcal{H}_{\mathcal{T}} \otimes \mathrm{TSym}^{k'} \mathcal{H}_{\mathcal{T}}(-j).$$

This Clebsch–Gordan map can also be realized geometrically using Lieberman’s trick, as follows. Recall the group $\mathfrak{T}_k = \mu_2^k \rtimes \mathfrak{S}_k$ and the character ε_k from §3.2 above. We saw that there are isomorphisms

$$H_{\mathcal{T}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathcal{T}}(n)) \cong H_{\mathcal{T}}^{i+k}(\mathcal{E}^k, \mathbf{Q}_{\mathcal{T}}(k+n))(\varepsilon_k),$$

and the same argument also gives isomorphisms

$$H_{\mathcal{T}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathcal{T}} \otimes \mathrm{TSym}^{k'} \mathcal{H}_{\mathcal{T}}(n)) \cong H_{\mathcal{T}}^{i+k+k'}(\mathcal{E}^{k+k'}, \mathbf{Q}_{\mathcal{T}}(k+k'+n))(\varepsilon_k \times \varepsilon_{k'}),$$

where we consider $\varepsilon_k \times \varepsilon_{k'}$ as a character of $\mathfrak{T}_k \times \mathfrak{T}_{k'} \subseteq \mathfrak{T}_{k+k'}$.

Lemma 5.2.1. *For any (k, k', j) satisfying the inequalities (5.1.1), we can find a finite set of triples $(\lambda_t, \xi_t, \eta_t)$, where $\lambda_t \in \mathbf{Q}$ and $\xi_t : \mathcal{E}^{k+k'-j} \rightarrow \mathcal{E}^{k+k'-2j}$ and $\eta_t : \mathcal{E}^{k+k'-2j} \rightarrow \mathcal{E}^{k+k'}$ are morphisms of Y -schemes, such that for any of the cohomology theories \mathcal{T} , the map*

$$H_{\mathcal{T}}^{k+k'-2j+i}(\mathcal{E}^{k+k'-2j}, \mathbf{Q}_{\mathcal{T}}(k+k'-2j+n)) \rightarrow H_{\mathcal{T}}^{k+k'+i}(\mathcal{E}^{k'}, \mathbf{Q}_{\mathcal{T}}(k+k'-j+n))$$

given by $\sum_t \lambda_t (\eta_t)_ \circ (\xi_t)^*$ sends the $\varepsilon_{k+k'-2j}$ -eigenspace to the $(\varepsilon_k \times \varepsilon_{k'})$ -eigenspace, and coincides on these eigenspaces with the map*

$$H_{\mathcal{T}}^i(Y, \mathrm{TSym}^{k+k'-2j} \mathcal{H}_{\mathcal{T}}(n)) \rightarrow H_{\mathcal{T}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathcal{T}} \otimes \mathrm{TSym}^{k'} \mathcal{H}_{\mathcal{T}}(n-j))$$

induced by $CG_{\mathcal{T}}^{[k,k',j]}$.

Proof. In all the cohomology theories \mathcal{T} we consider, for any morphism of Y -schemes $f : X \rightarrow X'$, there are relative pullback and (if f is proper) pushforward morphisms between the sheaves on Y obtained as the higher direct images of $\mathbf{Q}_{\mathcal{T}}$ along the structure maps of X and X' . These are compatible with the absolute pullback and pushforward via the Leray spectral sequence. So it suffices to show that we may find $(\lambda_t, \xi_t, \eta_t)$ such that the sum of the *relative* pushforward and pullback maps coincides with $CG_{\mathcal{T}}^{[k,k',j]}$. This then gives the result of the proposition (for all values of i and n simultaneously).

We consider first the extreme cases $j = 0$ and $k = k' = j$. In the former case, $CG_{\mathcal{T}}^{[k,k',0]}$ is just the natural inclusion $\mathrm{TSym}^{k+k'}(\mathcal{H}_{\mathcal{T}}) \subseteq \mathrm{TSym}^k(\mathcal{H}_{\mathcal{T}}) \otimes \mathrm{TSym}^{k'}(\mathcal{H}_{\mathcal{T}})$ (compatible with the inclusion of both sheaves into $(\mathcal{H}_{\mathcal{T}})^{\otimes(k+k')}$); so it is compatible with the natural inclusion of cohomology groups

$$H_{\mathcal{T}}^{k+k'+i}(\mathcal{E}^{k+k'}, \mathbf{Q}_{\mathcal{T}}(k+k'+n))(\varepsilon_{k+k'}) \rightarrow H_{\mathcal{T}}^{k+k'+i}(\mathcal{E}^{k'}, \mathbf{Q}_{\mathcal{T}}(k+k'+n))(\varepsilon_k \times \varepsilon_{k'}).$$

For the case $k = k' = j$, we note that pullback along the structure map $\pi_j : \mathcal{E}^j \rightarrow Y$, composed with pushforward along the diagonal inclusion $\delta_j : \mathcal{E}^j \rightarrow \mathcal{E}^{2j}$, gives a map of sheaves on Y (a “relative cycle class”)

$$\mathbf{Q}_{\mathcal{T}} \rightarrow R^{2j}(\pi_{2j})_* \mathbf{Q}_{\mathcal{T}}(j)$$

where π_{2j} is the projection $\mathcal{E}^{2j} \rightarrow Y$. Projecting to the subsheaf on which all the $[-1]$ endomorphisms on the fibres act as -1 , we obtain a map

$$\mathbf{Q}_{\mathcal{T}} \rightarrow (R^1 \pi_* \mathbf{Q}_{\mathcal{T}})^{\otimes 2j}(j) = \mathcal{H}_{\mathcal{T}}^{\otimes 2j}(-j).$$

Projecting to the direct summand $\mathrm{TSym}^{[j,j]} \mathcal{H}_{\mathcal{T}}(-j)$ gives a geometric realization of $CG_{\mathcal{T}}^{[j,j,j]}$, which is given concretely as a formal linear combination of translates of $(\delta_j)_*(\pi_j)^*$ by elements of the group $\mathfrak{T}_j \times \mathfrak{T}_j$.

For a general (k, k', j) , we take the product of the above maps for the triples (j, j, j) and $(k-j, k'-j, 0)$, and average over the cosets of $(\mathfrak{T}_j \times \mathfrak{T}_{k-j}) \times (\mathfrak{T}_j \times \mathfrak{T}_{k'-j})$ in $\mathfrak{T}_k \times \mathfrak{T}_{k'}$. Since the map $CG_{\mathcal{T}}^{[k,k',j]}$ is likewise built up from $CG_{\mathcal{T}}^{[j,j,j]}$ and $CG_{\mathcal{T}}^{[k-j,k'-j,0]}$ via the symmetrized tensor product, this gives the required compatibility. \square

Corollary 5.2.2. *There exists a morphism*

$$CG_{\mathrm{mot}}^{[k,k',j]} : H_{\mathrm{mot}}^i(Y, \mathrm{TSym}^{k+k'-2j} \mathcal{H}_{\mathbf{Q}}(n)) \rightarrow H_{\mathrm{mot}}^i(Y, \mathrm{TSym}^k \mathcal{H}_{\mathbf{Q}} \otimes \mathrm{TSym}^{k'} \mathcal{H}_{\mathbf{Q}}(n-j))$$

compatible with the maps $CG_{\mathcal{T}}^{[k,k',j]}$, for $\mathcal{T} \in \{B, \mathrm{dR}, \acute{\mathrm{e}}\mathrm{t}, \tilde{\acute{\mathrm{e}}}\mathrm{t}, \mathrm{rig}, \mathrm{syn}, \mathcal{H}\}$, under the regulator maps $r_{\mathcal{T}}$.

Proof. We simply define $CG_{\text{mot}}^{[k,k'],j]$ to be the map $\sum_t \lambda_t(\eta_t)_* \circ (\xi_t)^*$, which is well-defined since motivic cohomology with $\mathbf{Q}(n)$ coefficients has pullback and proper pushforward maps, compatible with the other theories via the maps $r_{\mathcal{T}}$. \square

We now suppose Y is a T -scheme, for some base scheme T , and we let Δ be the diagonal embedding $Y \hookrightarrow Y^2 = Y \times_T Y$.

Definition 5.2.3. We define sheaves on Y^2 by

$$\begin{aligned} \text{TSym}^{[k,k']} \mathcal{H}_{\mathcal{T}} &:= \pi_1^* \left(\text{TSym}^k \mathcal{H}_{\mathcal{T}} \right) \otimes \pi_2^* \left(\text{TSym}^{k'} \mathcal{H}_{\mathcal{T}} \right) \\ \text{Sym}^{(k,k')} \mathcal{H}_{\mathcal{T}} &:= \pi_1^* \left(\text{Sym}^k \mathcal{H}_{\mathcal{T}} \right) \otimes \pi_2^* \left(\text{Sym}^{k'} \mathcal{H}_{\mathcal{T}} \right), \end{aligned}$$

where π_1 and π_2 are the first and second projections $Y^2 \rightarrow Y$.

Then the pullback of these sheaves along Δ are the sheaves on Y

$$\begin{aligned} \Delta^* \text{TSym}^{[k,k']} \mathcal{H}_{\mathcal{T}} &\cong \text{TSym}^k \mathcal{H}_{\mathcal{T}} \otimes \text{TSym}^{k'} \mathcal{H}_{\mathcal{T}} \\ \Delta^* \text{Sym}^{(k,k')} \mathcal{H}_{\mathcal{T}} &\cong \text{Sym}^k \mathcal{H}_{\mathcal{T}} \otimes \text{Sym}^{k'} \mathcal{H}_{\mathcal{T}}. \end{aligned}$$

If Y is smooth of relative dimension d over T , then we obtain pushforward (Gysin) maps

$$\Delta_* : H_{\mathcal{T}}^i(Y, \text{TSym}^{[k,k']} \mathcal{H}_{\mathcal{T}}(n)) \rightarrow H_{\mathcal{T}}^{i+2d}(Y^2, \text{TSym}^{[k,k']} \mathcal{H}_{\mathcal{T}}(n+d))$$

for $\mathcal{T} \in \{B, \text{dR}, \text{ét}, \bar{\text{ét}}, \text{rig}, \text{syn}, \mathcal{H}\}$ (and any i and n). We extend this to the case $\mathcal{T} = \text{mot}$ by considering pushforward along the closed embedding of $\mathcal{E}^{k+k'} = \mathcal{E}^k \times_Y \mathcal{E}^{k'}$ into $\mathcal{E}^k \times_T \mathcal{E}^{k'}$.

5.3. Rankin–Eisenstein classes. We now come to the case which interests us: we consider the scheme $S = Y_1(N)$ over $T = \text{Spec } \mathbf{Z}[1/N]$.

Definition 5.3.1. For k, k', j satisfying the inequalities (5.1.1), and $\mathcal{T} \in \{\text{mot}, \text{ét}, \mathcal{H}, \text{syn}\}$, we define

$$\text{Eis}_{\mathcal{T}, b, N}^{[k,k'], j] := \left(\Delta_* \circ CG_{\mathcal{T}}^{[k,k'], j] \right) \left(\text{Eis}_{\mathcal{T}, b, N}^{k+k'-2j} \right) \in H_{\mathcal{T}}^3(Y_1(N)^2, \text{TSym}^{[k,k']}(\mathcal{H}_{\mathcal{T}})(2-j)).$$

Remark 5.3.2. The classes $\text{Eis}_{\mathcal{T}, b, N}^{[k,k'], j]$ can also be defined for the “geometric” theories $\mathcal{T} = \{\bar{\text{ét}}, \text{dR}, B, \text{rig}\}$, but these are automatically zero, since $Y_1(N)^2$ is an affine surface, and thus its H^3 vanishes.

5.4. Abel–Jacobi maps. Let f, g be newforms of weights $k+2, k'+2$ and levels N_f, N_g dividing N . The aim of this section is the construction of Abel–Jacobi maps $\text{AJ}_{\mathcal{T}, f, g}$, for $\mathcal{T} \in \{\mathcal{H}, \text{syn}, \text{ét}\}$, which we will use to interpret the Rankin–Eisenstein classes as linear functionals on de Rham cohomology.

Absolute Hodge cohomology. Since $Y_1(N)^2$ is a smooth affine variety of dimension 2, its de Rham (or Betti) cohomology vanishes in degrees ≥ 3 . Consequently, the exact sequence (2.3.1) for absolute Hodge cohomology gives an isomorphism

$$H_{\mathcal{H}}^3(Y_1(N)_{\mathbf{R}}^2, \text{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{R}})(2-j)) \cong H_{\mathcal{H}}^1(\text{Spec } \mathbf{R}, H_B^2(Y_1(N)_{\mathbf{R}}^2, \text{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{R}})(2-j)))$$

for any j . Via projection to the (f, g) -isotypical component we obtain a natural map

$$H_{\mathcal{H}}^3(Y_1(N)_{\mathbf{R}}^2, \text{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{R}})(2-j)) \longrightarrow H_{\mathcal{H}}^1(\text{Spec } \mathbf{R}, M_B(f \otimes g)^*(-j)).$$

The comparison isomorphisms between Betti and de Rham cohomology induce two *period maps*

$$\begin{aligned} \alpha_{M(f \otimes g)(n)} : M_B(f \otimes g)(n)_{\mathbf{R}}^+ &\rightarrow t(M(f \otimes g)(n))_{\mathbf{R}} \\ \alpha_{M(f \otimes g)^*(n)} : M_B(f \otimes g)^*(n)_{\mathbf{R}}^+ &\rightarrow t(M(f \otimes g)^*(n))_{\mathbf{R}} \end{aligned}$$

with the tangent spaces from 3.4.4 The L -vector spaces $M_B(f \otimes g)(n)^+$ and $M_B(f \otimes g)^*(n)^+$ have dimension 2 and in the case that $0 \leq j \leq \min\{k, k'\}$ we get that $\ker(\alpha_{M(f \otimes g)(j+1)})$ is one-dimensional. Poincaré duality induces a perfect pairing

$$\ker(\alpha_{M(f \otimes g)(j+1)}) \times \text{coker}(\alpha_{M(f \otimes g)^*(-j)}) \rightarrow L \otimes_{\mathbf{Q}} \mathbf{R}$$

which together with the isomorphism coming from sequence (2.3.2)

$$H_{\mathcal{H}}^1(\text{Spec } \mathbf{R}, M_B(f \otimes g)^*(-j)) \cong \text{coker}(\alpha_{M(f \otimes g)^*(-j)})$$

induces the isomorphism

$$H_{\mathcal{H}}^1(\text{Spec } \mathbf{R}, M_B(f \otimes g)^*(-j)) \cong \ker(\alpha_{M(f \otimes g)(j+1)})^*.$$

Putting these isomorphisms together, we obtain the *Abel–Jacobi map* for absolute Hodge cohomology,

$$\mathrm{AJ}_{\mathcal{H},f,g} : H_{\mathcal{H}}^3(Y_1(N)_{\mathbf{R}}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{R}})(2-j)) \longrightarrow (\ker \alpha_{M(f \otimes g)(1+j)})^*.$$

Taking duals we get an exact sequence

$$(5.4.1) \quad 0 \rightarrow \mathrm{Fil}^{-j} M_{\mathrm{dR}}(f \otimes g)_{\mathbf{R}}^* \rightarrow M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^+ \rightarrow H_{\mathcal{H}}^1(\mathrm{Spec} \mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}}) \rightarrow 0,$$

which is crucial for the interpretation of the leading terms of $L(f, g, s)$ at $s = j + 1$ in the Beilinson conjecture.

Syntomic cohomology. Similarly, for a prime $p \nmid N$, the exact sequence (2.3.4) for syntomic cohomology gives isomorphisms

$$H_{\mathrm{syn}}^3(Y_1(N)_{\mathbf{Z}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \cong H_{\mathrm{syn}}^1(\mathrm{Spec} \mathbf{Z}_p, H_{\mathrm{rig}}^2(Y_1(N)_{\mathbf{Z}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathrm{rig}})(2-j))),$$

and projecting to the (f, g) -isotypical component we obtain a map

$$\begin{aligned} H_{\mathrm{syn}}^3(Y_1(N)_{\mathbf{Z}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) &\longrightarrow H_{\mathrm{syn}}^1(\mathrm{Spec} \mathbf{Z}_p, M_{\mathrm{rig}}(f \otimes g)^*(-j)) \\ &= \frac{M_{\mathrm{rig}}(f \otimes g)^*(-j)_{\mathbf{Q}_p}}{(1-\varphi) \mathrm{Fil}^0 M_{\mathrm{dR}}(f \otimes g)^*(-j)_{\mathbf{Q}_p}}. \end{aligned}$$

Since $1 - \varphi$ is an isomorphism on $M_{\mathrm{rig}}(f \otimes g)^*(-j)_{\mathbf{Q}_p}$ for $0 \leq j \leq \min(k, k')$, the right-hand side can be identified with $t(M(f \otimes g)^*(-j))_{\mathbf{Q}_p}$, which is free of rank 3 over $L \otimes \mathbf{Q}_p$. This gives the *syntomic Abel–Jacobi map*

$$\mathrm{AJ}_{\mathrm{syn},f,g} : H_{\mathrm{syn}}^3(Y_1(N)_{\mathbf{Z}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \longrightarrow t(M(f \otimes g)^*(-j))_{\mathbf{Q}_p}.$$

Étale cohomology. The étale spectral sequence ${}^{\mathrm{ét}}E^{ij}$ for $Y_1(N)_{\mathbf{Q}_p}^2$ degenerates at E_3 (since $H^i(\mathbf{Q}_p, -)$ is the zero functor for $i \neq \{0, 1, 2\}$), so we obtain a natural map (not an isomorphism in general)

$$H_{\mathrm{ét}}^3(Y_1(N)_{\mathbf{Q}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \rightarrow H_{\mathrm{ét}}^1(\mathrm{Spec} \mathbf{Q}_p, H_{\mathrm{ét}}^2(Y_1(N)_{\mathbf{Q}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j))).$$

Projecting to the (f, g) -isotypical component gives an étale Abel–Jacobi map

$$\mathrm{AJ}_{\mathrm{ét},f,g} : H_{\mathrm{ét}}^3(Y_1(N)_{\mathbf{Q}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \longrightarrow H^1(\mathbf{Q}_p, M_{\mathrm{ét}}(f \otimes g)^*(-j)).$$

The following proposition gives a relation between the syntomic and étale Abel–Jacobi maps:

Proposition 5.4.1. *The maps $\mathrm{AJ}_{\mathrm{ét},f,g}$ and $\mathrm{AJ}_{\mathrm{syn},f,g}$ fit into a commutative diagram*

$$\begin{array}{ccc} H_{\mathrm{mot}}^3(Y_1(N)_{\mathbf{Z}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) & \xrightarrow{\quad} & H_{\mathrm{mot}}^3(Y_1(N)_{\mathbf{Q}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \\ \downarrow r_{\mathrm{syn}} & & \downarrow r_{\mathrm{ét}} \\ H_{\mathrm{syn}}^3(Y_1(N)_{\mathbf{Z}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) & & H_{\mathrm{ét}}^3(Y_1(N)_{\mathbf{Q}_p}^2, \mathrm{TSym}^{[k,k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \\ \downarrow \mathrm{AJ}_{\mathrm{syn},f,g} & & \downarrow \mathrm{AJ}_{\mathrm{ét},f,g} \\ t(M(f \otimes g)^*(-j))_{\mathbf{Q}_p} & \xrightarrow{\quad \exp \circ \mathrm{comp}_{\mathrm{dR}} \quad} & H^1(\mathbf{Q}_p, M_{\mathrm{ét}}(f \otimes g)^*(-j)). \end{array}$$

where the top horizontal arrow is given by base-extension, and in the bottom horizontal arrow,

$$\mathrm{comp}_{\mathrm{dR}} : M_{\mathrm{dR}}(f \otimes g)_{\mathbf{Q}_p}^* \cong \mathbf{D}_{\mathrm{dR}}(M_{\mathrm{ét}}(f \otimes g)^*)$$

is the Faltings comparison isomorphism, and \exp is the Bloch–Kato exponential map (c.f. Section 2.4).

Proof. This follows from Theorem 2.4.2(1) applied to the variety $X = \mathcal{E}^k \times \mathcal{E}^{k'}$, together with the observation that the (f, g) -eigenspaces in de Rham, syntomic, and étale cohomology all lift isomorphically to the cohomology of the product of Kuga–Sato varieties $\overline{\mathcal{E}}^k \times \overline{\mathcal{E}}^{k'}$, which is projective, so we may apply Theorem 2.4.2(2) (with $X = \overline{\mathcal{E}}^k \times \overline{\mathcal{E}}^{k'}$). \square

We shall give formulae for the images of the Eisenstein class under $\mathrm{AJ}_{\mathcal{H},f,g}$ and $\mathrm{AJ}_{\mathrm{syn},f,g}$ in the following section, and by the preceding proposition, the latter will also give a formula for $\mathrm{AJ}_{\mathrm{ét},f,g}$. (We will not use $\mathrm{AJ}_{\mathrm{ét},f,g}$ directly in the present paper, but it will play a central role in the sequel.)

6. REGULATOR FORMULAE

6.1. Differentials and rationality. Recall that we have fixed newforms f, g of levels N_f, N_g and weights $k+2, k'+2$. We let $\omega_f \in H^0(X_1(N)(\mathbf{C}), \text{Sym}^k \mathcal{H}_{\mathbf{C}}^{\vee} \otimes \Omega^1) \otimes_{\mathbf{Q}} L$ denote the holomorphic $(\text{Sym}^k \mathcal{H}_{\mathbf{C}}^{\vee})$ -valued differential whose pullback to the upper half-plane is given by

$$\omega_f = (2\pi i)^{k+1} f(\tau) w^{(k,0)} d\tau = (2\pi i)^k f(\tau) w^{(k,0)} \frac{dq}{q},$$

where as in §4.4, we have $w^{(k,0)} = (dz)^k$, for dz the standard basis of $\text{Fil}^1 \mathcal{H}_{\mathbf{C}}^{\vee}$.

Lemma 6.1.1. *Let $f = \sum_{n>0} a_n q^n$ with $a_n \in L$ and ε_f the associated character. We denote by N_{ε_f} its conductor. Let*

$$G(\varepsilon_f^{-1}) := \sum_{x \in \mathbf{Z}/N_{\varepsilon_f} \mathbf{Z}} \varepsilon_f^{-1}(x) e^{2\pi i x / N_{\varepsilon_f}}$$

be the Gauss sum of ε_f^{-1} . Then the differential

$$\omega'_f := G(\varepsilon_f^{-1}) \omega_f$$

is a section of $\text{Sym}^k \mathcal{H}_{\text{dR}}^{\vee}$ over L , and its class in de Rham cohomology is a basis of the 1-dimensional L -vector space $\text{Fil}^1 M_{\text{dR}}(f)$.

Proof. Note that the cusp ∞ is not defined over \mathbf{Q} . By the q -expansion principle, we have to check that for any $\sigma_d \in \text{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \cong (\mathbf{Z}/N\mathbf{Z})^*$, we have $\sigma_d(a_n) = a_n$. But the action of σ_d is given by $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^* f = \varepsilon_f(d) f$, which implies that the coefficients a_n have to be in the ε -eigenspace of $L \otimes_{\mathbf{Q}} \mathbf{Q}(\zeta_N)$, which is generated over L by $G(\varepsilon_f^{-1})$. \square

We also want to have a basis of the space $M(f)/\text{Fil}^1$, for which we need to introduce the form conjugate to f :

Definition 6.1.2. *We define $f^* \in S_{k+2}(\Gamma_1(N), \mathbf{C})$ to be the form with q -expansion $\sum_{n>0} \overline{a_n} q^n$, so that*

$$f^*(\tau) = \overline{f(-\bar{\tau})}.$$

The class of the C^∞ differential form

$$\bar{\omega}_{f^*} = (-1)^{k+1} f(-\bar{\tau}) w^{(0,k)} d\bar{\tau}$$

has the same Hecke eigenvalues as f , so it defines an element of $M_{\text{dR}}(f) \otimes_{\mathbf{Q}} \mathbf{C}$.

Proposition 6.1.3. *Let $\langle -, - \rangle_{Y_1(N_f)}$ denote the Poincaré duality pairing. Then the class modulo Fil^1 of the element*

$$\frac{G(\varepsilon_f^{-1})}{\langle \omega_f, \bar{\omega}_{f^*} \rangle_{Y_1(N_f)}} \bar{\omega}_{f^*}$$

is non-zero and defined over L , and thus defines a basis η'_f of the L -vector space $\frac{M_{\text{dR}}(f)}{\text{Fil}^1 M_{\text{dR}}(f)}$.

Proof. It suffices to check that this differential pairs to an element of L with the basis vector $\omega'_{f^*} = G(\varepsilon_f) \omega_{f^*}$ of $\text{Fil}^1 M_{\text{dR}}(f^*)$. Since $\langle \omega_{f^*}, \bar{\omega}_{f^*} \rangle_{Y_1(N_f)} = \langle \omega_f, \bar{\omega}_f \rangle_{Y_1(N_f)}$, this pairing evaluates to $G(\varepsilon_f) G(\varepsilon_f^{-1}) = (-1)^k N_{\varepsilon_f} \in L$, as required. \square

Remark 6.1.4. (1) The constant $\langle \omega_f, \bar{\omega}_f \rangle_{Y_1(N_f)}$ is equal to $(-4\pi)^{k+1} k! \|f\|$, where

$$\|f\| = \int_{\Gamma_1(N_f) \backslash \mathfrak{H}} |f(x+iy)|^2 y^k dx dy$$

is the Petersson norm of f .

- (2) In [LLZ14] we worked directly with the classes ω_f and $\eta_f = \frac{1}{\langle \omega_f, \bar{\omega}_f \rangle} \bar{\omega}_{f^*}$. However, these classes are not defined over L but only over $L \otimes \mathbf{Q}(\mu_N)$ for a suitable N ; so one has to extend scalars to this field in order to evaluate the regulators. (This base-extension was inadvertently omitted from the statement of Theorem 5.4.6 of *op.cit.*.) In the present paper we want to verify the conjectures of Beilinson and Perrin-Riou, which predict the values of regulators up to a factor in L^\times , so it is more convenient to work with the L -rational classes η'_f and ω'_f .

6.2. The regulator formula in absolute Hodge cohomology. Let $0 \leq j \leq \min\{k, k'\}$ and write $Y := Y_1(N)_{\mathbf{R}}$ and $Y^2 := Y_1(N)_{\mathbf{R}} \times Y_1(N)_{\mathbf{R}}$. As in §5.4, let f, g be newforms of weights $k+2, k'+2$ and levels N_f, N_g dividing N . In this section we want to give a formula for the image of $\text{Eis}_{\mathcal{H}, b, N}^{[k, k', j]}$ under the Abel–Jacobi map

$$\text{AJ}_{\mathcal{H}, f, g} : H_{\mathcal{H}}^3(Y_{\mathbf{R}}^2, \text{TSym}^{[k, k']} \mathcal{H}_{\mathbf{R}}(2-j)) \longrightarrow (\ker \alpha_{M(f \otimes g)(1+j)})^*.$$

By 5.4.1 one has an exact sequence

$$0 \rightarrow F^{-j} M_{\text{dR}}(f \otimes g)_{\mathbf{R}}^* \rightarrow M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^+ \rightarrow (\ker \alpha_{M(f \otimes g)(1+j)})^* \rightarrow 0$$

and we will compute a representative for $\text{Eis}_{\mathcal{H}, b, N}^{[k, k', j]}$ in $M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^+$. To compute the projection $\text{pr}_{f, g}$ we use the perfect pairing

$$\langle \quad, \quad \rangle_{Y^2} : M_B(f \otimes g)_{\mathbf{C}}^* / F^{-j} \times F^{1+j} M_B(f \otimes g)_{\mathbf{C}} \rightarrow \mathbf{C} \otimes_{\mathbf{Q}} L.$$

Remark 6.2.1. This pairing is in fact the product of the pairings

$$\begin{aligned} \langle \quad, \quad \rangle_Y : M_B(f)_{\mathbf{C}}^* \times M_B(f)_{\mathbf{C}} &\rightarrow \mathbf{C} \otimes_{\mathbf{Q}} L, \\ \langle \quad, \quad \rangle_Y : M_B(g)_{\mathbf{C}}^* \times M_B(g)_{\mathbf{C}} &\rightarrow \mathbf{C} \otimes_{\mathbf{Q}} L. \end{aligned}$$

We want to give a formula for the pairing of

$$\text{Eis}_{\mathcal{H}, b, N}^{[k, k', j]} := \left(\Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]} \right) \left(\text{Eis}_{\mathcal{H}, b, N}^{k+k'-2j} \right)$$

with a class in $F^{1+j} M_B(f \otimes g)_{\mathbf{C}}^+$ which reduces the computation to an integral on $Y(\mathbf{C})$. For this we consider $\text{Eis}_{\mathcal{H}, b, N}^{[k, k', j]}$ as a class in $H_B^2(Y^2(\mathbf{C}), \text{TSym}^{[k, k']} \mathcal{H}_{\mathbf{C}})^+ / F^{2-j}$. Consider the pull-back Δ^* composed with the dual of $CG_B^{[k, k', j]}$,

$$\vee CG_B^{[k, k', j]} \circ \Delta^* : F^{1+j} H_{c, B}^2(Y^2(\mathbf{C}), \text{Sym}^{(k, k')} \mathcal{H}_{\mathbf{C}}^{\vee}) \rightarrow F^{1+j} H_{c, B}^2(Y(\mathbf{C}), \text{Sym}^{k+k'-2j} \mathcal{H}_{\mathbf{C}}^{\vee}).$$

Recall from Proposition 4.4.5 that $\text{Eis}_{\mathcal{H}, b, N}^{k+k'-2j}$ is represented by a pair of forms $(\alpha_{\infty}, \alpha_{\text{dR}})$. Then we have:

Proposition 6.2.2. *For any cohomology class $[\omega_c] \in F^{1+j} M_{\text{dR}}(f \otimes g)_{\mathbf{R}}$ one has the formula*

$$\langle \text{Eis}_{\mathcal{H}, b, N}^{[k, k', j]}, \omega_c \rangle_{Y^2} = \frac{1}{2\pi i} \int_{Y(\mathbf{C})} \vee CG_B^{[k, k', j]} \circ \Delta^*(\omega_c) \wedge \alpha_{\infty}.$$

Remark 6.2.3. Compare [LLZ14, Theorem 4.3.1], which is the case of trivial coefficients. Note that the proof will actually show that the integral is well defined, i.e., does not depend on the choice of a differential ω_c representing the class $[\omega_c]$.

Proof. The push-forward along the diagonal Δ is defined via Deligne homology (see [Jan88b] for the definitions). In fact one has by general properties for a Bloch–Ogus cohomology theory

$$H_{\mathcal{H}}^1(Y_{\mathbf{R}}, \text{TSym}^{k+k'-2j} \mathcal{H}_{\mathbf{R}}(1)) \cong H_1^{\mathcal{H}}(Y_{\mathbf{R}}, \text{Sym}^{k+k'-2j} \mathcal{H}_{\mathbf{R}}^{\vee})$$

and

$$H_{\mathcal{H}}^3(Y_{\mathbf{R}}^2, \text{TSym}^{[k, k']} \mathcal{H}_{\mathbf{R}}(2-j)) \cong H_1^{\mathcal{H}}(Y_{\mathbf{R}}^2, \text{Sym}^{(k, k')} \mathcal{H}_{\mathbf{R}}^{\vee}(j)).$$

With these isomorphisms the map $\Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}$ is just the functoriality for the homology combined with the Clebsch–Gordan map. These homology groups have an interpretation in terms of currents. Let $T_{\alpha_{\infty}}$ and $T_{\alpha_{\text{dR}}}$ be the currents associated to α_{∞} and α_{dR} . As $H_{\text{dR}}^3(Y_{\mathbf{R}}^2, \text{TSym}^{[k, k']} \mathcal{H}_{\mathbf{R}}) = 0$ the current $\Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}(T_{\alpha_{\text{dR}}})$ is a trivial cohomology class, so that there exists a current ρ (with logarithmic singularities) with $d\rho = \Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}(T_{\alpha_{\text{dR}}})$. It follows that $\Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}(T_{\alpha_{\infty}}) - \rho$ defines a cohomology class, which represents a lift of $\text{Eis}_{\mathcal{H}, b, N}^{[k, k', j]}$ to $H_B^2(Y^2(\mathbf{C}), \text{TSym}^{[k, k']} \mathcal{H}_{\mathbf{C}})$. This gives

$$\langle \text{Eis}_{\mathcal{H}, b, N}^{[k, k', j]}, \omega_c \rangle_{Y^2} = (\Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}(T_{\alpha_{\infty}}) - \rho)(\omega_c).$$

By construction $\Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}(T_{\alpha_{\text{dR}}})$ is a current in the zeroth step of the Hodge filtration (see [Jan88b, 1.4]) so that also ρ is in F^0 . As ω_c is in $F^{1+j} M_B(f \otimes g)_{\mathbf{C}}$ the evaluation $\rho(\omega_c)$ is in F^{1+j} and hence zero as $1+j > 0$. This gives

$$(\Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}(T_{\alpha_{\infty}}) - \rho)(\omega_c) = \Delta_* \circ CG_{\mathcal{H}}^{[k, k', j]}(T_{\alpha_{\infty}})(\omega_c) = T_{\alpha_{\infty}}(\vee CG_B^{[k, k', j]} \circ \Delta^*(\omega_c))$$

where the last equality is the definition of the push-forward. Finally, by definition of T_{α_∞} , we get

$$T_{\alpha_\infty}(\vee CG_B^{[k,k',j]} \circ \Delta^*(\omega_c)) = \frac{1}{2\pi i} \int_{Y(\mathbf{C})} \vee CG_B^{[k,k',j]} \circ \Delta^*(\omega_c) \wedge \alpha_\infty.$$

As all these equalities hold for any closed form ω_c in the $1+j$ -step of the Hodge filtration and because $\langle \text{Eis}_{\mathcal{H},b,N}^{[k,k',j]}, \omega_c \rangle_B$ is independent of the representative of $[\omega_c]$, the same is true for the integral. \square

For the explicit computations we use the bases

$$\{\omega_f \otimes \omega_g, \omega_f \otimes \bar{\omega}_{g^*}, \bar{\omega}_{f^*} \otimes \omega_g, \bar{\omega}_{f^*} \otimes \bar{\omega}_{g^*}\} \quad \text{and} \quad \{\omega_{f^*} \otimes \omega_{g^*}, \omega_{f^*} \otimes \bar{\omega}_g, \bar{\omega}_f \otimes \omega_{g^*}, \bar{\omega}_f \otimes \bar{\omega}_g\}$$

of $M_B(f \otimes g)_{\mathbf{C}}$ and $M_B(f^* \otimes g^*)_{\mathbf{C}}$ respectively.

Remark 6.2.4. The natural map $M_B(f^* \otimes g^*)(k+k'+2) \rightarrow M_B(f \otimes g)^*$ is an isomorphism, so we may interpret the latter vectors as a basis of $M_B(f \otimes g)^*_{\mathbf{C}}$.

Note that one has $\bar{F}_\infty^*(\omega_{f^*} \otimes \omega_{g^*}) = \bar{\omega}_f \otimes \bar{\omega}_g$ and $\bar{F}_\infty^*(\omega_{f^*} \otimes \bar{\omega}_g) = \bar{\omega}_f \otimes \omega_{g^*}$ so that $\frac{1}{2}(\omega_{f^*} \otimes \bar{\omega}_g + (-1)^{-j-1} \bar{\omega}_f \otimes \omega_{g^*})$ is a basis of

$$M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^{+/F^{-j}} M_{\text{dR}}(f \otimes g)^*_{\mathbf{R}} \cong (\ker \alpha_{M(f \otimes g)(1+j)})^*.$$

Lemma 6.2.5. *The element*

$$\frac{-1}{\langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} (\bar{\omega}_{f^*} \otimes \omega_g + (-1)^{j+1} \omega_f \otimes \bar{\omega}_{g^*})$$

in $\ker \alpha_{M(f \otimes g)(1+j)}$ is the dual basis of $\frac{1}{2}(\omega_{f^} \otimes \bar{\omega}_g + (-1)^{-j-1} \bar{\omega}_f \otimes \omega_{g^*})$.*

Proof. This follows from the formulae

$$\begin{aligned} \langle \bar{\omega}_{f^*} \otimes \omega_g, \omega_{f^*} \otimes \bar{\omega}_g \rangle_Y &= -\langle \omega_{f^*}, \bar{\omega}_{f^*} \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y \\ \langle \omega_f \otimes \bar{\omega}_{g^*}, \bar{\omega}_f \otimes \omega_{g^*} \rangle_Y &= -\langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_{g^*}, \bar{\omega}_{g^*} \rangle_Y, \end{aligned}$$

as $\langle \omega_{f^*}, \bar{\omega}_{f^*} \rangle_Y = \langle \omega_f, \bar{\omega}_f \rangle_Y$ and $\langle \omega_g, \bar{\omega}_g \rangle_Y = \langle \omega_{g^*}, \bar{\omega}_{g^*} \rangle_Y$. \square

Definition 6.2.6. *We write*

$$\bar{\omega}_{f^*} \wedge \omega_g \wedge \alpha_\infty$$

for the form on $Y(\mathbf{C})$ obtained from the form $\vee CG_B^{[k,k',j]} \circ \Delta^(\bar{\omega}_{f^*} \otimes \omega_g) \wedge \alpha_\infty$ by using the evaluation pairing $\text{TSym}^{k+k'-2j} \mathcal{H}_{\mathbf{C}} \otimes \text{Sym}^{k+k'-2j} \mathcal{H}_{\mathbf{C}}^\vee \rightarrow \mathbf{C}$ and similar for $\omega_f \wedge \bar{\omega}_{g^*} \wedge \alpha_\infty$.*

Proposition 6.2.7. *Let $0 \leq j \leq \min\{k, k'\}$ and $\text{Eis}_{\mathcal{H},b,N}^{k+k'-2j} = (\alpha_\infty, \alpha_{\text{dR}})$. Then*

$$\begin{aligned} \left\langle \text{AJ}_{\mathcal{H},f,g} \left(\text{Eis}_{\mathcal{H},b,N}^{[k,k',j]} \right), \frac{-1}{\langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} (\bar{\omega}_{f^*} \otimes \omega_g + (-1)^{j+1} \omega_f \otimes \bar{\omega}_{g^*}) \right\rangle = \\ \frac{-1}{(2\pi i) \langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} \int_{Y(\mathbf{C})} (\bar{\omega}_{f^*} \wedge \omega_g + (-1)^{j+1} \omega_f \wedge \bar{\omega}_{g^*}) \wedge \alpha_\infty. \end{aligned}$$

Proof. We have to compute the pairing of $\text{AJ}_{\mathcal{H},f,g} \left(\text{Eis}_{\mathcal{H},b,N}^{[k,k',j]} \right)$ with

$$\frac{-1}{\langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} (\bar{\omega}_{f^*} \otimes \omega_g + (-1)^{j+1} \omega_f \otimes \bar{\omega}_{g^*})$$

which by Proposition 6.2.2 reduces to the integral in the proposition. \square

We will compute the integral appearing in Proposition 6.2.7 in terms of special values of the L -function $L(f, g, s)$. We write

$$\omega_{f^*} = f^*(2\pi i)^k w^{(k,0)} \frac{dq}{q} \quad \text{and} \quad \omega_g = g(2\pi i)^{k'} w^{(k',0)} \frac{dq}{q}$$

so that

$$\bar{\omega}_{f^*} \wedge \omega_g = (-1)^{k+1} \bar{f}^* g(2\pi i)^{k+k'+2} w^{(0,k)} \otimes w^{(k',0)} d\tau d\bar{\tau}.$$

Recall from Proposition 4.4.5 the formula

$$\begin{aligned} \alpha_\infty &:= \frac{-N^{k+k'-2j}}{2} \times \\ &\sum_{m=0}^{k+k'-2j} (-1)^m (k+k'-2j-m)! (2\pi i)^{m-k-k'+2j} (\tau - \bar{\tau})^m F_{2m-k-k'+2j, k+k'-2j+1-m, b}^{\text{an}} w^{[k+k'-2j-m, m]} \end{aligned}$$

Proposition 6.2.8. *One has*

$$\begin{aligned}\bar{\omega}_{f*} \wedge \omega_g \wedge \alpha_\infty &= \frac{(-1)^j N^{k+k'-2j}}{2} \binom{k}{j} k'! (2\pi i)^{k+2} (\tau - \bar{\tau})^k \bar{f}^* g F_{k-k', k'-j+1, b}^{\text{an}} d\tau d\bar{\tau} \\ \omega_f \wedge \bar{\omega}_{g*} \wedge \alpha_\infty &= \frac{(-1)^{k+k'+1} N^{k+k'-2j}}{2} \binom{k'}{j} k! (2\pi i)^{k'+2} (\tau - \bar{\tau})^{k'} f \bar{g}^* F_{k'-k, k-j+1, b}^{\text{an}} d\tau d\bar{\tau}\end{aligned}$$

Proof. From Proposition 5.1.2 one sees that

$$\vee CG_B^{[k, k', j]}(w^{(0, k)} \otimes w^{(k', 0)}) \wedge w^{[s, t]} = w^{(0, k)} \otimes w^{(k', 0)} \wedge CG_B^{[k, k', j]}(w^{[s, t]})$$

is zero unless $(s, t) = (k' - j, k - j)$, in which case one gets

$$\frac{k!(k')!}{j!(k-j)!(k'-j)!} \left(\frac{\tau - \bar{\tau}}{2\pi i} \right)^j.$$

Collecting terms gives the first formula. The second formula is obtained in a similar way, observing that

$$w^{(0, k)} \otimes w^{(k', 0)} \wedge CG_B^{[k, k', j]}(w^{[s, t]}) = (-1)^{s+t-j} w^{(k, 0)} \otimes w^{(0, k')} \wedge CG_B^{[k, k', j]}(w^{[t, s]}). \quad \square$$

Theorem 6.2.9. *Let $0 \leq j \leq \min\{k, k'\}$ and $b = 1$. One has*

$$\begin{aligned}\frac{1}{2\pi i} \int_{Y(\mathbf{C})} \bar{\omega}_{f*} \wedge \omega_g \wedge \alpha_\infty &= \frac{(-1)^{k-j}}{2} (2\pi i)^{k+k'-2j} \frac{k!k'!}{(k-j)!(k'-j)!} L'(f, g, j+1) \\ \frac{1}{2\pi i} \int_{Y(\mathbf{C})} \omega_f \wedge \bar{\omega}_{g*} \wedge \alpha_\infty &= \frac{(-1)^{k-1}}{2} (2\pi i)^{k+k'-2j} \frac{k!k'!}{(k-j)!(k'-j)!} L'(f, g, j+1),\end{aligned}$$

where $L'(f, g, j+1) := \lim_{s \rightarrow 0} \frac{L(f, g, j+1+s)}{s}$. In particular,

$$\begin{aligned}\left\langle \text{AJ}_{\mathcal{H}, f, g} \left(\text{Eis}_{\mathcal{H}, 1, N}^{[k, k', j]} \right), \frac{-1}{\langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} (\bar{\omega}_{f*} \otimes \omega_g + (-1)^{j+1} \omega_f \otimes \bar{\omega}_{g*}) \right\rangle = \\ \frac{(-1)^{k-j+1} (2\pi i)^{k+k'-2j}}{2 \langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} \frac{k!k'!}{(k-j)!(k'-j)!} L'(f, g, j+1).\end{aligned}$$

Proof. This follows from the formulae above, the functional equation

$$F_{k-k', k'-j+1, 1}^{\text{an}} = E_{1/N}^{(k-k')}(\tau, j-k)$$

in the notation of [LLZ14, Definition 4.2.1] and the formulae in (3.5.1) with $r = k+2$, $r' = k'+2$. \square

6.3. Finite-polynomial cohomology. We now compute the image of the Eisenstein class under the p -adic syntomic Abel–Jacobi map, following [BDR14a] and [DR14]. In order that we can apply Besser’s rigid syntomic cohomology, we need to assume that $p \nmid N$.

We fix integers (k, k', j) satisfying our usual inequalities (5.1.1). Let $Y = Y_1(N)_{\mathbf{Z}_p}$ and $S = Y \times_{\text{Spec } \mathbf{Z}_p} Y$, $Y_{\mathbf{Q}_p}$ and $S_{\mathbf{Q}_p}$ their generic fibres, and \mathcal{Y} and \mathcal{S} the smooth pairs (Y, X) and $(Y \times Y, X \times X)$, where $X = X_1(N)_{\mathbf{Z}_p}$.

Let \mathcal{F} be the sheaf $\text{TSym}^{[k, k']} \mathcal{H}_{\mathbf{Q}_p} := \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p} \boxtimes \text{TSym}^{k'} \mathcal{H}_{\mathbf{Q}_p}$ on S (regarded as an overconvergent filtered F -isocrystal), and \mathcal{F}^\vee its dual.

As we saw in Section 5.4 above, there is a natural map

$$\text{AJ}_{\text{syn}, f, g} : H_{\text{mot}}^3(S, \mathcal{F}(2-j)) \longrightarrow t(M(f \otimes g)^*(-j))_{\mathbf{Q}_p} = (\text{Fil}^{1+j} M_{\text{dR}}(f \otimes g)_{\mathbf{Q}_p})^*.$$

Our inequalities on j imply that $\text{Fil}^{1+j} M_{\text{dR}}(f \otimes g)$ is 3-dimensional over L .

Suppose we are given an element

$$\lambda \in \text{Fil}^0 M_{\text{dR}}(f \otimes g)(1+j)_{\mathbf{Q}_p}.$$

We want to evaluate the pairing

$$\left\langle \text{AJ}_{\text{syn}, f, g} \left(\text{Eis}_{\text{syn}, b, N}^{[k, k', j]} \right), \lambda \right\rangle.$$

Since $1 - p^{-1}\varphi^{-1}$ is an isomorphism on $M_{\text{dR}}(f \otimes g)(1+j)$, we may find a polynomial $P \in 1 + TL_{\mathfrak{P}}[T]$ such that $P(p^{-1}) \neq 0$ and $P(\varphi)(\lambda) = 0$. Thus λ lifts to a class

$$\tilde{\lambda} \in H_{\text{fp}, c}^2(S, \mathcal{F}^\vee(1+j), P);$$

and since $H_{\text{dR}, c}^1(S, \mathcal{F}^\vee(1+j))$ is zero (being Poincaré dual to H^3 with non-compact supports) this lift is uniquely determined.

Proposition 6.3.1. *We have*

$$\left\langle \mathrm{AJ}_{\mathrm{syn},f,g} \left(\mathrm{Eis}_{\mathrm{syn},b,N}^{[k,k',j]} \right), \lambda \right\rangle = \left\langle \mathrm{Eis}_{\mathrm{syn},b,N}^{[k,k',j]}, \tilde{\lambda} \right\rangle_{\mathrm{fp},S},$$

where the pairing $\langle -, - \rangle_{\mathrm{fp},S}$ is as defined in §2.5 above.

Proof. This is an instance of the general statement that the cup-product on finite-polynomial cohomology is compatible with the Leray spectral sequence, as observed in §2.5. \square

Corollary 6.3.2. *We have*

$$\left\langle \mathrm{AJ}_{\mathrm{syn},f,g} \left(\mathrm{Eis}_{\mathrm{syn},b,N}^{[k,k',j]} \right), \lambda \right\rangle = \left\langle CG_{\mathrm{syn}}^{[k,k',j]}(\mathrm{Eis}_{\mathrm{syn},b,N}^{k+k'-2j}), \Delta^*(\tilde{\lambda}) \right\rangle_{\mathrm{fp},Y}.$$

Proof. Recall that $\mathrm{Eis}_{\mathrm{syn},b,N}^{[k,k',j]} = (\Delta_* \circ CG_{\mathrm{syn}}^{[k,k',j]})(\mathrm{Eis}_{\mathrm{syn},b,N}^{k+k'-2j})$. The pushforward in fp-cohomology satisfies the projection formula ([Bes12, Theorem 5.2]), so

$$\left\langle \mathrm{Eis}_{\mathrm{syn},b,N}^{[k,k',j]}, \tilde{\lambda} \right\rangle_{\mathrm{fp},S} = \left\langle CG_{\mathrm{syn}}^{[k,k',j]}(\mathrm{Eis}_{\mathrm{syn},b,N}^{k+k'-2j}), \Delta^*(\tilde{\lambda}) \right\rangle_{\mathrm{fp},Y}. \quad \square$$

This reduces the computation of the syntomic regulator to a calculation on Y alone. We now give a more explicit recipe when λ is of a special form. We suppose that

$$\lambda = \eta \sqcup \omega := \pi_1^*(\eta) \cup \pi_2^*(\omega),$$

where π_1, π_2 are the two projections $S \rightarrow Y$, and $\eta \in \mathrm{Fil}^0 M_{\mathrm{dR}}(f)_{\mathbf{Q}_p}$, $\omega \in \mathrm{Fil}^0 M_{\mathrm{dR}}(g)_{\mathbf{Q}_p}(1+j)$. We fix polynomials P_η (pure of weight $k+1$) and P_ω (pure of weight $k'+1-2j$) annihilating η and ω respectively. Then η and ω lift uniquely to classes

$$\begin{aligned} \tilde{\eta} &\in H_{\mathrm{fp},c}^1(Y, \mathrm{Sym}^k \mathcal{H}_{\mathbf{Q}_p}^\vee, P_\eta), \\ \tilde{\omega} &\in H_{\mathrm{fp},c}^1(Y, \mathrm{Sym}^k \mathcal{H}_{\mathbf{Q}_p}^\vee(1+j), P_\omega) \end{aligned}$$

and we clearly have $\tilde{\lambda} = \tilde{\eta} \sqcup \tilde{\omega}$ and hence

$$\Delta^*(\tilde{\lambda}) = \Delta^*(\tilde{\eta} \sqcup \tilde{\omega}) = \tilde{\eta} \cup \tilde{\omega}.$$

Thus we have

$$\left\langle CG_{\mathrm{syn}}^{[k,k',j]}(\mathrm{Eis}_{\mathrm{syn},b,N}^{k+k'-2j}), \Delta^*(\tilde{\lambda}) \right\rangle_{\mathrm{fp},Y} = \mathrm{tr}_{\mathrm{fp},Y}(\sigma \cup \tilde{\eta} \cup \tilde{\omega}),$$

where we have written $\sigma = CG_{\mathrm{syn}}^{[k,k',j]}(\mathrm{Eis}_{\mathrm{syn},b,N}^{k+k'-2j})$ for brevity, and $\mathrm{tr}_{\mathrm{fp},Y}$ is as in §2.5 above.

Proposition 6.3.3. *The natural map*

$$H_{\mathrm{rig}}^1(Y, \mathrm{TSym}^k \mathcal{H}_{\mathrm{rig}}(2)) \rightarrow H_{\mathrm{fp}}^2(\mathcal{Y}, \mathrm{TSym}^k \mathcal{H}(2), P_\omega)$$

is surjective; and if $\Xi \in H_{\mathrm{rig}}^1(Y, \mathrm{TSym}^k \mathcal{H}_{\mathrm{rig}}(2))$ is any preimage of $\sigma \cup \tilde{\omega}$, then we have

$$\left\langle CG_{\mathrm{syn}}^{[k,k',j]}(\mathrm{Eis}_{\mathrm{syn},b,N}^{k+k'-2j}), \tilde{\eta} \cup \tilde{\omega} \right\rangle_{\mathrm{fp},Y} = \langle P_\omega(p^{-1}\varphi^{-1})^{-1}\eta, \Xi \rangle_{\mathrm{rig},Y}.$$

Proof. This natural map is clearly surjective (because the cokernel is a subspace of $H_{\mathrm{rig}}^2(\mathcal{Y}, \mathrm{TSym}^k \mathcal{H}_{\mathrm{rig}}(2))$, which is zero because Y is affine). The equality of cup-products is another instance of the compatibility of finite-polynomial cup-products with the Leray spectral sequence, as noted above. \square

In the next section we'll compute such a lifting Ξ explicitly.

6.4. Restriction to the ordinary locus. As in Section 4.5, we denote by Y^{ord} the open subscheme of Y where the Eisenstein series E_{p-1} is invertible, so $\mathcal{Y}^{\mathrm{ord}} = (Y^{\mathrm{ord}}, X)$ is also a smooth pair. Note that $Y_{\mathbf{Q}_p}^{\mathrm{ord}}$ is the complement of a finite set of points in $Y_{\mathbf{Q}_p}$, possibly defined over the unramified quadratic extension of \mathbf{Q}_p (one for each supersingular point of the special fibre).

Proposition 6.4.1. *The rigid realization $M_{\mathrm{rig}}(f)$ lifts canonically to a subspace of $H_{\mathrm{rig},c}^1(Y^{\mathrm{ord}}, \mathrm{Sym}^k \mathcal{H}_{\mathrm{rig}})$, commuting with the action of φ .*

Proof. This follows from the fact that the systems of Hecke eigenvalues appearing in the cokernel of the natural map

$$H_{\mathrm{rig},c}^1(Y^{\mathrm{ord}}, \mathrm{Sym}^k \mathcal{H}_{\mathrm{rig}}) \rightarrow H_{\mathrm{rig},c}^1(Y, \mathrm{Sym}^k \mathcal{H}_{\mathrm{rig}})$$

are those associated to p -new cusp forms of level $\Gamma_1(N) \cap \Gamma_0(p)$, which are disjoint from those appearing in level $\Gamma_1(N)$. \square

We let η^{ord} be the image of η under this lifting, and let Ξ be as in Proposition 6.3.3. Then we have

$$\langle P_\omega(p^{-1}\varphi^{-1})^{-1}\eta, \xi \rangle_{\text{rig}, Y} = \langle P_\omega(p^{-1}\varphi^{-1})^{-1}\eta^{\text{ord}}, \Xi|_{Y^{\text{ord}}} \rangle_{\text{rig}, Y^{\text{ord}}}.$$

We now compute a representative for $\Xi|_{Y^{\text{ord}}}$.

Definition 6.4.2. Choose a ∇ -closed algebraic section of $\Omega^1 \otimes \text{Fil}^0 \mathcal{H}^\vee(1+j)$ over $Y_{\mathbf{Q}_p}$ representing the class of ω (which we shall denote, abusively, by the same letter ω), and let

$$F_\omega \in \Gamma\left(X_{\mathbf{Q}_p}^{\text{rig}}, j^\dagger \mathcal{H}_{\mathbf{Q}_p}^\vee(1+j)|_{Y^{\text{ord}}}\right)$$

be a rigid-analytic primitive of $P_\omega(\varphi)(\omega)$, so that the class of the pair (ω, F_ω) is a lift of ω to $H_{\text{fp}}^1(\mathcal{Y}^{\text{ord}}, \mathcal{H}_{\mathbf{Q}_p}^\vee(1+j), P_\omega)$.

As we saw above, the restriction to Y^{ord} of the syntomic Eisenstein class $\text{Eis}_{\text{syn}, b, N}^{k+k'-2j}$ is represented by the pair $(\alpha_{\text{rig}}, \alpha_{\text{dR}})$ defined in Theorem 4.5.7 above, satisfying $\nabla(\alpha_{\text{rig}}) = (1-\varphi)\alpha_{\text{dR}}$. Let

$$\begin{aligned}\sigma_{\text{rig}} &= CG_{\text{rig}}^{[k, k', j]}(\alpha_{\text{rig}}), \\ \sigma_{\text{dR}} &= CG_{\text{dR}}^{[k, k', j]}(\alpha_{\text{dR}}),\end{aligned}$$

so the restriction to Y^{ord} of the class $CG_{\text{syn}}^{[k, k', j]}(\text{Eis}_{\text{syn}, b, N}^{k+k'-2j})$ is given by the pair $(\sigma_{\text{rig}}, \sigma_{\text{dR}})$. The definition of the cup-product in finite-polynomial cohomology now gives the following:

Proposition 6.4.3. The class $\Xi|_{Y^{\text{ord}}}$ is represented by the $\text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(2)$ -valued overconvergent 1-form on Y^{ord} defined by

$$\Xi := \bigcup [a(\varphi_1, \varphi_2)(F_\omega \otimes \sigma_{\text{dR}}) - b(\varphi_1, \varphi_2)(\omega \otimes \sigma_{\text{rig}})],$$

where $a(x, y)$ and $b(x, y)$ are any two polynomials such that

$$P_\omega(xy) = a(x, y)P_\omega(x) + b(x, y)(1-y). \quad \square$$

Note that such polynomials do exist, since the polynomial $P(X) = 1 - X$ is the identity for the \star operation of Definition 2.5.3. The form Ξ is evidently ∇ -closed (since Y is 1-dimensional) and hence defines a class in $H_{\text{rig}}^1(Y^{\text{ord}}, \text{TSym}^k \mathcal{H}_{\mathbf{Q}_p}(2))$; and this class is well-defined, since changing the polynomials (a, b) changes P_ω by an exact form.

We now evaluate the right-hand side of the formula in Proposition 6.3.3 in terms of p -adic modular forms. We take for ω the class $\omega'_g(1+j)$, where $\omega'_g \in \text{Fil}^{1+j} M_{\text{dR}}(g)$ is as in 6.1; and we take for P_ω the polynomial

$$P_g(X) = \left(1 - \frac{p^{1+j}X}{\alpha_g}\right) \left(1 - \frac{p^{1+j}X}{\beta_g}\right)$$

where α_g, β_g are the roots of the Hecke polynomial of g at p .

We denote also by ω'_g the unique regular algebraic differential on $X_1(N_f)$ representing this class, whose pullback to the formal scheme \tilde{Y} of §4.5 is given by

$$\omega'_g = G(\varepsilon_g^{-1})g v^{(k', 0)} \otimes \xi \otimes e_{1+j}.$$

Remark 6.4.4. Note that this class ω'_g does not generally have q -expansion in $L[[q]]$, owing to the presence of the Gauss sum, but it is nonetheless defined over L ; recall that the cusp ∞ is not rational on our model of $Y_1(N)$.

Remark 6.4.5. More generally, one can replace g with the any holomorphic form \check{g} of level N having q -expansion in $(L \otimes \mathbf{Q}_p)[[q]]$ and the same Hecke eigenvalues as g away from N . We give the argument for $\check{g} = g$ for simplicity of notation.

As a rigid 1-form we have

$$P_g(\varphi)(\omega'_g) = G(\varepsilon_g^{-1})g^{[p]} v^{(k', 0)} \otimes \xi \otimes e_{1+j},$$

where $g^{[p]}$ is the “ p -depletion” of g (the p -adic modular form whose q -expansion is $\sum_{p \nmid n} a_n(g)q^n$). If we write G for an overconvergent primitive of $P_g(\varphi)(\omega'_g)$ vanishing at the cusp ∞ , then as above we have

$$\Xi|_{Y^{\text{ord}}} = \bigcup [a(\varphi_1, \varphi_2)(G \otimes \sigma_{\text{dR}}) - b(\varphi_1, \varphi_2)(\omega'_g \otimes \sigma_{\text{rig}})],$$

Here $a(X, Y)$ and $b(X, Y)$ may be any polynomials such that $P_g(XY) = a(X, Y)P_g(X) + b(X, Y)(1 - Y)$, but we shall make the following choice

$$a(X, Y) = 1, \quad b(X, Y) = \frac{P_g(XY) - P_g(X)}{1 - Y},$$

so that

$$\Xi|_{Y^{\text{ord}}} = G \cup \sigma_{\text{dR}} - \bigcup [b(\varphi_1, \varphi_2)(\omega'_g \otimes \sigma_{\text{rig}})].$$

Lemma 6.4.6. *If G vanishes at the cusp ∞ , then modulo ∇ -exact 1-forms we have*

$$G \cup \sigma_{\text{dR}} = G \cup (1 - \varphi)\sigma_{\text{dR}} = G \cup \nabla(\sigma_{\text{rig}}).$$

Proof. It suffices to show that $G \cup \varphi(\sigma_{\text{dR}})$ is ∇ -exact. We know that the q -expansion of G is p -depleted (the coefficient of q^n is zero if $p \mid n$) while $\varphi(\sigma_{\text{dR}})$ is a power series in q^p . Hence the q -expansion of $G \cup \varphi(\sigma_{\text{dR}})$ is also p -depleted, and thus lies in the kernel of the operator U_p defined on q -expansions by $U_p(\sum a_n q^n) = \sum a_{np} q^n$. Since $U_p \circ \varphi$ coincides with the diamond operator $\langle p \rangle$, and $\langle p \rangle$ and φ act bijectively on the cohomology groups, any differential in the kernel of U_p must be ∇ -exact. \square

As a corollary, we obtain the following explicit formula for $\Xi|_{Y^{\text{ord}}}$:

Corollary 6.4.7. *Modulo ∇ -exact 1-forms, we have*

$$(6.4.1) \quad \Xi|_{Y^{\text{ord}}} = - \left(1 - \frac{p^{k+2}\varphi^2}{\langle p \rangle} \right) (\omega'_g \otimes \sigma_{\text{rig}}).$$

Proof. Observe that the 1-form

$$G \cup \nabla(\sigma_{\text{rig}}) + \nabla(G) \cup \sigma_{\text{rig}} = \nabla(G \cup \sigma_{\text{rig}})$$

is ∇ -exact. Hence modulo ∇ -exact forms we have

$$G \cup \nabla(\sigma_{\text{rig}}) = -\nabla(G) \cup \sigma_{\text{rig}} = -\bigcup [P_g(\varphi_1)(\omega'_g \otimes \sigma_{\text{rig}})].$$

Thus

$$\Xi|_{Y^{\text{ord}}} = -\bigcup [c(\varphi_1, \varphi_2)(\omega'_g \otimes \sigma_{\text{rig}})]$$

where

$$c(X, Y) = P_g(X) + b(X, Y) = \frac{P_g(XY) - Y P_g(X)}{1 - Y}.$$

We evaluate explicitly:

$$c(X, Y) = 1 - \frac{p^{2+2j} X^2 Y}{\alpha_g \beta_g}.$$

Hence

$$\Xi|_{Y^{\text{ord}}} = \omega'_g \cup \sigma_{\text{rig}} - \frac{p^{2+2j}}{\alpha_g \beta_g} \varphi(\varphi(\omega'_g) \cup \sigma_{\text{rig}}).$$

We now note that the relation

$$\sigma_{\text{rig}} = p^{1+k+k'-2j} \langle p \rangle^{-1} \varphi(\sigma_{\text{rig}})$$

holds modulo exact forms (as one sees by an explicit q -expansion calculation). Hence

$$\Xi|_{Y^{\text{ord}}} = - \left(1 - \frac{p^{k+k'+3}\varphi^2}{\alpha_g \beta_g \langle p \rangle} \right) (\langle p \rangle \omega'_g \otimes \sigma_{\text{rig}}) = - \left(1 - \frac{p^{k+2}\varphi^2}{\langle p \rangle} \right) (\omega'_g \otimes \sigma_{\text{rig}}). \quad \square$$

6.5. Relation to a p -adic L -value. Recall that above η was an arbitrary element of the free rank 2 $(L \otimes \mathbf{Q}_p)$ -module

$$\text{Fil}^0 M_{\text{dR}}(f)_{\mathbf{Q}_p} = M_{\text{dR}}(f)_{\mathbf{Q}_p}.$$

We now specify η more precisely. Recall that in §6.1 we defined a canonical L -basis η'_f of $M_{\text{dR}}(f)/\text{Fil}^1$.

Choose a prime \mathfrak{P} of L above p , and (after extending L if necessary) choose a root $\alpha_f \in L_{\mathfrak{P}}$ of the Hecke polynomial $X^2 - a_p(f)X + p^{k+1}\varepsilon_p(f)$ of f . Assume that α has p -adic valuation $< k+1$ (i.e. it is not the non-unit root associated to an ordinary form). Then the $\varphi = \alpha$ eigenspace in $M_{\text{rig}}(f) \otimes_{L \otimes \mathbf{Q}_p} L_{\mathfrak{P}}$ is complementary to the Fil^1 subspace.

Definition 6.5.1. *We let η_f^α be the unique lifting of $\eta'_f \in M_{\text{dR}}(f)/\text{Fil}^1$ to an element of $M_{\text{dR}}(f) \otimes_L L_{\mathfrak{P}}$ satisfying $\varphi(\eta_f^\alpha) = \alpha_f \eta_f^\alpha$.*

Remark 6.5.2. If f is ordinary (so α_f is necessarily the unit root) the class η_f^α coincides with $G(\varepsilon_f^{-1})\eta_f^{\text{ur}}$ where η_f^{ur} is the class considered in [DR14, Proposition 4.6], [LLZ14, Theorem 5.6.2].

We may lift η_f^α uniquely to a compactly-supported class $\eta_f^{\alpha, \text{ord}}$ in the rigid cohomology of Y^{ord} and having the same Hecke eigenvalues outside p .

Remark 6.5.3. The data implicit in a lifting of η_f^α to the compactly-supported cohomology of Y^{ord} is a choice of local integrals of η_f^α on some sufficiently small annulus along the boundary of each supersingular residue disc.

Proposition 6.5.4. *We have*

$$\langle \eta_f^\alpha, \Xi \rangle_{\text{rig}, Y} = - \left(1 - \frac{\beta_f}{p\alpha_f} \right) \langle \eta_f^{\alpha, \text{ord}}, \omega'_g \cup \sigma_{\text{rig}} \rangle_{\text{rig}, Y^{\text{ord}}},$$

where $\beta_f = p^{k+1}\varepsilon_f(p)/\alpha_f$ is the other root of the Hecke polynomial of f .

Proof. This follows from Equation (6.4.1), since we have

$$\begin{aligned} \langle \eta_f^\alpha, \Xi \rangle_{\text{rig}, Y} &= \langle \eta_f^{\alpha, \text{ord}}, \Xi|_{Y^{\text{ord}}} \rangle_{\text{rig}, Y^{\text{ord}}} \\ &= - \left\langle \eta_f^{\alpha, \text{ord}}, \left(1 - \frac{p^{k+2}\varphi^2}{\langle p \rangle} \right) (\omega'_g \cup \sigma_{\text{rig}}) \right\rangle_{\text{rig}, Y^{\text{ord}}} \\ &= - \left\langle \left(1 - \frac{p^{k+2}(p^{-1}\varphi^{-1})^2}{\langle p^{-1} \rangle} \right) \eta_f^{\alpha, \text{ord}}, \omega'_g \cup \sigma_{\text{rig}} \right\rangle_{\text{rig}, Y^{\text{ord}}} \\ &= - \left(1 - \frac{\beta_f}{p\alpha_f} \right) \langle \eta_f^{\alpha, \text{ord}}, \omega'_g \cup \sigma_{\text{rig}} \rangle_{\text{rig}, Y^{\text{ord}}}. \quad \square \end{aligned}$$

To proceed further we need a more explicit description of the class $\omega'_g \cup \sigma_{\text{rig}}$. It turns out that we only need to consider the image of $\omega'_g \cup \sigma_{\text{rig}}$ under a certain projection operator:

Definition 6.5.5 (cf. [DR14, §2.4]). *The unit root splitting of $\mathcal{H}_{\text{rig}}|_{Y^{\text{ord}}}$ is the map*

$$\text{spl}^{\text{ur}} : \mathcal{H}_{\text{rig}}|_{Y^{\text{ord}}} \rightarrow \text{Fil}^0 \mathcal{H}_{\text{rig}}|_{Y^{\text{ord}}}$$

whose kernel is the unit root subspace for the action of Frobenius.

Proposition 6.5.6. *The image of $\omega'_g \cup \sigma_{\text{rig}}$ under spl^{ur} is the p -adic modular form*

$$-N^{k+k'-2j}(-1)^{k'-j-1}(k')! \binom{k}{j} G(\varepsilon_g^{-1}) \left(g \cdot F_{k-k', k'-j+1, b}^{(p)} \right).$$

Proof. Recall that $\sigma_{\text{rig}} = CG_{\text{syn}}^{[k, k', j]}(\alpha_{\text{rig}})$, and the pullback of α_{rig} to the formal scheme \mathcal{Y} of §4.5 is given by the sum

$$\alpha_{\text{rig}} = -N^{k+k'-2j} \sum_{i=0}^{k+k'-2j} (-1)^{k+k'-2j-i} (k+k'-2j-i)! F_{2i-(k+k'-2j), k+k'-2j+1-i, b}^{(p)} v^{[k+k'-2j-i, i]}.$$

In terms of the basis $\{v^{[r, s]} : r+s = k'\}$ of $\text{TSym}^k \mathcal{H}$, the unit-root splitting spl^{ur} sends all basis vectors to zero except $v^{[k, 0]}$. Hence, by the definition of the trilinear pairing and Proposition 5.1.2, we see that the linear map given by pairing with $G(\varepsilon_g^{-1})g \otimes v^{(k', 0)}$ and applying spl^{ur} sends all terms in this sum to zero except the term for $i = k-j$; this term is given by

$$-N^{k+k'-2j}(-1)^{k'-j-1}(k'-j)! F_{k-k', k'-j+1, b}^{(p)} v^{[k'-j, k-j]},$$

and Proposition 5.1.2 gives the factor $\frac{k!(k')!}{j!(k-j)!(k'-j)!}$, which gives the claimed formula. \square

The restriction of the unit-root splitting to the overconvergent $(\text{TSym}^k \mathcal{H})$ -valued differentials on Y^{ord} is injective, and (after tensoring with $\mathbf{Q}(\mu_N)$ to rectify issues with rationality of cusps) its image is the space $M_{k+2}^{\text{n-oc}}(N, L_{\mathfrak{P}})$ of “nearly overconvergent” p -adic modular forms, in the sense of [DR14, Definition 2.6].

There is a map, the *overconvergent projector*,

$$\Pi^{\text{oc}} : M_{k+2}^{\text{n-oc}}(N, L_{\mathfrak{P}}) \rightarrow \frac{M_{k+2}^{\dagger}(N, L_{\mathfrak{P}})}{\theta^{k+1} M_{-k}^{\dagger}(N, L_{\mathfrak{P}})}$$

(cf. [DR14, Equation (44)]). Composing this with the unit-root splitting gives an isomorphism

$$H_{\text{rig}}^1(Y^{\text{ord}}, \text{TSym}^k \mathcal{H}_{\text{rig}}) \otimes \mathbf{Q}(\mu_N) \cong \frac{M_{k+2}^\dagger(N, L_{\mathfrak{P}})}{\theta^{k+1} M_{-k}^\dagger(N, L_{\mathfrak{P}})} \otimes \mathbf{Q}(\mu_N).$$

All slopes of the U_p operator acting on the denominator $\theta^{k+1} M_{-k}^\dagger(N, L_{\mathfrak{P}})$ are $\geq k+1$, so pairing with $\eta_f^{\alpha, \text{ord}}$ annihilates this space. The quotient is isomorphic to the classical forms of level $\Gamma_1(N(p)) = \Gamma_1(N) \cap \Gamma_0(p)$, and we obtain the following corollary:

Corollary 6.5.7. *We have*

$$\langle \eta_f^\alpha, \Xi \rangle_{\text{rig}, Y} = (-1)^{k'-j+1} N^{k+k'-2j} (k')! \binom{k}{j} G(\varepsilon_g^{-1}) \left(1 - \frac{\beta_f}{p\alpha_f}\right) \left\langle \eta_f^{\alpha, \text{ord}}, \Pi^{\text{oc}} \left(g \cdot F_{k-k', k'-j+1, b}^{(p)}\right) \right\rangle_{N(p)},$$

and hence

$$\begin{aligned} \left\langle \text{AJ}_{\text{syn}, f, g} \left(\text{Eis}_{\text{syn}, b, N}^{[k, k', j]} \right), \eta_f^\alpha \otimes \omega'_g \right\rangle &= \frac{1}{P_g(p^{-1} \alpha_f^{-1})} \langle \eta_f^{\alpha, \text{ord}}, \Xi \rangle_{\text{rig}, Y} \\ &= (-1)^{k'-j+1} N^{k+k'-2j} (k')! \binom{k}{j} G(\varepsilon_g^{-1}) \frac{\left(1 - \frac{\beta_f}{p\alpha_f}\right)}{\left(1 - \frac{p^j}{\alpha_f \alpha_g}\right) \left(1 - \frac{p^j}{\alpha_f \beta_g}\right)} \\ &\quad \times \left\langle \eta_f^{\alpha, \text{ord}}, \Pi^{\text{oc}} \left(g \cdot F_{k-k', k'-j+1, b}^{(p)}\right) \right\rangle_{N(p)}. \end{aligned}$$

In order to make the link to p -adic L -functions, we would like a version of this with the Eisenstein series $F_{t, s, b}^{(p)}$ replaced with the “ p -depleted” one

$$F_{t, s, b}^{[p]} = \sum_{\substack{n \geq 0 \\ p \nmid n}} q^n \sum_{dd'=n} d^{t-1+s} (d')^{-s} (\zeta_N^{bd'} + (-1)^t \zeta_N^{-bd'}).$$

This is the specialization of a 2-variable p -adic analytic family, with t, s varying over characters of \mathbf{Z}_p^\times . It is easy to see that $F_{k-k', k'-j+1, b}^{(p)}$ is an eigenvector for $U = U_p$ with eigenvalue p^{k-j} , and evidently

$$(1 - VU) F_{k-k', k'-j+1, b}^{(p)} = F_{k-k', k'-j+1, b}^{[p]}$$

where V is the right inverse of U given by $q \mapsto q^p$ on $\mathbf{Z}[[q]]$. We also have the following power-series identity:

Lemma 6.5.8. *Let R be a commutative ring and let $A, B \in R[[q]]$ be such that $UA = \lambda A - \mu VA$ and $UB = \nu B$. Then we have*

$$A \cdot (1 - VU)B = (1 - \lambda\nu V + \mu\nu^2 V^2) (A \cdot B) + (1 - VU) [\mu V^2(A)B - AV(B)].$$

□

Applying the lemma with $A = g$ and $B = F_{k-k', k'-j+1, 1}^{(p)}$, and noting that anything in the image of $1 - VU$ is annihilated under projection to a finite-slope U_p -eigenspace, we can rearrange Corollary 6.5.7 to give the following formula: if $\mathcal{E}(f, g, 1+j) \neq 0$ then

$$(6.5.1) \quad \left\langle \text{AJ}_{\text{syn}, f, g} \left(\text{Eis}_{\text{syn}, b}^{[k, k', j]} \right), \eta_f^\alpha \otimes \omega'_g \right\rangle = (-1)^{k'-j+1} N^{k+k'-2j} (k')! \binom{k}{j} G(\varepsilon_g^{-1}) \frac{\mathcal{E}(f)}{\mathcal{E}(f, g, 1+j)} \times \left\langle \eta_f^{\alpha, \text{ord}}, \Pi^{\text{oc}} \left(g \cdot F_{k-k', k'-j+1, b}^{[p]}\right) \right\rangle_{N(p)},$$

where $\mathcal{E}(f)$ and $\mathcal{E}(f, g, s)$ are as defined in Theorem 3.5.3 (and we assume $\mathcal{E}(f, g, 1+j)$ does not vanish).

Theorem 6.5.9. *Suppose that f is ordinary, with α_f be the unit root. If $\mathcal{E}(f, g, 1+j) \neq 0$ then we have*

$$\left\langle \text{AJ}_{\text{syn}, f, g} \left(\text{Eis}_{\text{syn}, 1, N}^{[k, k', j]} \right), \eta_f^\alpha \otimes \omega'_g \right\rangle = (-1)^{k'-j+1} (k')! \binom{k}{j} G(\varepsilon_f^{-1}) G(\varepsilon_g^{-1}) \frac{\mathcal{E}(f) \mathcal{E}^*(f)}{\mathcal{E}(f, g, 1+j)} L_p(f, g, 1+j),$$

where the p -adic L -function $L_p(f, g, 1+j)$ and the factors $\mathcal{E}(f)$ and $\mathcal{E}^*(f)$ are as defined in Theorem 3.5.3.

Proof. Since α_f is the unit root, pairing with η_f^α factors through the Hida ordinary idempotent e_{ord} ; and we have $e_{\text{ord}}\phi = e_{\text{ord}}(\Pi^{\text{oc}}\phi)$ for any nearly-overconvergent form ϕ [DR14, Lemma 2.7]. By the construction of the p -adic L -function, we have

$$N^{k+k'-2j} \left\langle \eta_f^{\alpha, \text{ord}}, e_{\text{ord}} \left(g \cdot F_{k-k', k'-j+1, 1}^{[p]} \right) \right\rangle_{N(p)} = G(\varepsilon_f^{-1}) \mathcal{E}^*(f) L_p(f, g, 1+j).$$

(See [LLZ14, Proposition 5.4.1]; the power of N and the Gauss sum appear because our normalizations are slightly different from *op.cit.*, see Remark 6.5.2 and Remark 3.5.4(1) above). Substituting this into equation (6.5.1) gives the stated formula. \square

Remark 6.5.10.

- (i) The non-vanishing of $\mathcal{E}(f, g, 1+j)$ is automatic, for weight reasons, unless $k = k' = j$.
- (ii) The p -adic Eisenstein series $F_{k-k', k'-j+1, 1}^{[p]}$ is the same as the one denoted by $\mathcal{E}_{b/N}(j - k' + 1, k - j)$ in [LLZ14, Definition 5.3.1].
- (iii) This argument works without the ordinary assumption on f , if we use Urban's definition of the p -adic Rankin–Selberg L -function [Urb14]. This construction is only written up for $N = 1$, but the extension to general N is immediate.
- (iv) The factors $\mathcal{E}(f)$ and $\mathcal{E}^*(f)$ can be interpreted as Euler factors attached to the adjoint L -function of f , which measures the difference between the period $\langle \omega_f, \bar{\omega}_f \rangle$ used in defining η'_f and a “correctly normalized” period.

7. THE PERRIN-RIOU CONJECTURE

7.1. The Beilinson conjecture. In this section we consider $0 \leq j \leq \min\{k, k'\}$ and let f and g be new forms of weights k and k' and levels N_f, N_g dividing N , respectively. We want to prove part of Beilinson's conjecture for $L(f, g, s)$ at $s = j + 1$.

With our conditions on j , it follows from the functional equation that $L(f, g, s)$ has a zero of order 1 at $s = j + 1$. Denote by $L'(f, g, j + 1)$ the value of the derivative of $L(f, g, s)$ at $j + 1$. Beilinson conjectures the following interpretation of this value:

One expects that there is a Chow motive $M(f \otimes g)^*$ underlying the realizations discussed so far in this paper, whose motivic cohomology $H_{\text{mot}}^1(\mathbf{Q}, M(f \otimes g)^*(-j))$ should be a direct summand of $H_{\text{mot}}^3(Y_1(N)^2, \text{TSym}^{[k, k']} \mathcal{H}(2 - j))$ defined by the action of the Hecke algebra. Unfortunately, the existence of the Chow motive $M(f \otimes g)^*$ is only known in the case of $k = k' = 0$, i.e., if f and g have weight 2. Beilinson conjectures further the existence of a subspace of integral elements

$$H_{\text{mot}}^1(\mathbf{Z}, M(f \otimes g)^*(-j)) \subset H_{\text{mot}}^1(\mathbf{Q}, M(f \otimes g)^*(-j))$$

of the motivic cohomology of $M(f \otimes g)^*$, which should have L -dimension 1 and such that the regulator induces an isomorphism

$$r_{\mathcal{H}} : H_{\text{mot}}^1(\mathbf{Z}, M(f \otimes g)^*(-j)) \otimes_{\mathbf{Q}} \mathbf{R} \cong H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}}).$$

In our results, which will be formulated below, we do not have to say anything about the dimension of the motivic cohomology. Recall from 5.4.1 the exact sequence

$$(7.1.1) \quad 0 \rightarrow \text{Fil}^{-j} M_{\text{dR}}(f \otimes g)^*_{\mathbf{R}} \rightarrow M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^+ \rightarrow H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}}) \rightarrow 0$$

and the isomorphism $H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}}) \cong (\ker \alpha_{M(f \otimes g)(j+1)})^*$. The exact sequence induces an isomorphism

$$\det_{\mathbf{R} \otimes L}(M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^+) \otimes \det_{\mathbf{R} \otimes L}(\text{Fil}^{-j} M_{\text{dR}}(f \otimes g)^*_{\mathbf{R}})^{-1} \cong \det_{\mathbf{R} \otimes L}(H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}})).$$

Beilinson defines an L -structure on the left hand side as follows:

Conjecture 7.1.1 (Beilinson). *Denote by*

$$\mathcal{R}(M(f \otimes g)^*(-j)) := \det_L(M_B(f \otimes g)^*(-j-1)) \otimes \det_L(\text{Fil}^{-j} M_{\text{dR}}(f \otimes g)^*)$$

the one-dimensional L -vector space of $\det_{\mathbf{R} \otimes L}(H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^(-j)_{\mathbf{R}}))$. Then*

$$r_{\mathcal{H}}(H_{\text{mot}}^1(\mathbf{Z}, M(f \otimes g)^*(-j))) = L'(f, g, j+1) \mathcal{R}(M(f \otimes g)^*(-j)).$$

Recall from 5.3.1 the Rankin–Eisenstein class

$$\mathrm{AJ}_{\mathcal{H},f,g}(\mathrm{Eis}_{\mathcal{H},1,N}^{[k,k',j]}) \in H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}}),$$

which is by construction in the image of the regulator map

$$r_{\mathcal{H}} : H_{\mathrm{mot}}^1(\mathbf{Q}, M(f \otimes g)^*(-j)) \otimes_{\mathbf{Q}} \mathbf{R} \rightarrow H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}}).$$

Remark 7.1.2. Scholl has defined in [Sch00] a subspace of integral elements in motivic cohomology, which coincides with Beilinson’s integral elements if the variety admits a projective, flat and regular model. We remark that the motivic Eisenstein class $\mathrm{Eis}_{\mathrm{mot},1,N}^{k+k'-2j}$ can be defined integrally over the regular model $Y_1(N)_{\mathbf{Z}}$ over \mathbf{Z} by considering the moduli space of elliptic curves with a point of exact order N in the sense of Drinfeld. With the techniques in loc. cit. one can show that $\mathrm{Eis}_{\mathrm{mot},1,N}^{[k,k',j]}$ lies in the subspace of Scholl’s integral elements.

Before we formulate our results, we choose bases to make the involved L -structures more explicit. Recall that f^* and g^* denote the forms with complex conjugate Fourier coefficients and that we have associated differential forms $\omega_f = f(\tau)(2\pi)^{k+1}w^{(k,0)}d\tau$.

Observe that $\varepsilon_{f^*} = \varepsilon_f^{-1}$ and that $\overline{G(\varepsilon_f)} = \varepsilon_f(-1)G(\varepsilon_f^{-1}) = (-1)^{k+2}G(\varepsilon_f^{-1})$. Moreover one has $G(\varepsilon_f)\overline{G(\varepsilon_f)} = N_{\varepsilon_f}$. Let $M(\varepsilon_f)$ the Artin motive associated to the character ε_f . The cup-product between cohomology in degree 0 and 1 induces an isomorphism of motives

$$(7.1.2) \quad M(\varepsilon_f)(k+1) \otimes M(f) \cong M(f^*)(k+1) \cong M(f)^*.$$

Definition 7.1.3. Denote by $\omega_{\varepsilon_f} \in M_{\mathrm{dR}}(\varepsilon_f)$ and $\delta_{\varepsilon_f} \in M_B(\varepsilon_f)$ generators such that $\omega_{\varepsilon_f} = G(\varepsilon_f)\delta_{\varepsilon_f}$. This is possible by [Del79, Section 6.4].

Note 7.1.4. We have $(2\pi i)^{k+1}\delta_{\varepsilon_f} \in M_B(\varepsilon_f)(k+1)^-$ because ε_f has parity $k+2$.

Definition 7.1.5. Choose a basis δ_f^{\pm} of $M_B(f)^{\pm}$ and δ_g^{\pm} of $M_B(g)^{\pm}$, so that

$$\{\delta_f^+ \otimes \delta_g^+, \delta_f^+ \otimes \delta_g^-, \delta_f^- \otimes \delta_g^+, \delta_f^- \otimes \delta_g^-\}$$

is an L -basis of $M_B(f \otimes g)$.

Definition 7.1.6. Let

$$\begin{aligned} \delta_{f^*}^{\pm} &:= (2\pi i)^{k+1}\delta_{\varepsilon_f}\delta_f^{\mp} \in M_B(f^*)(k+1)^{\pm} \\ \delta_{g^*}^{\pm} &:= (2\pi i)^{k'+1}\delta_{\varepsilon_g}\delta_g^{\mp} \in M_B(g^*)(k'+1)^{\pm}. \end{aligned}$$

We normalize δ_{ε_f} and δ_{ε_g} such that $\langle \delta_{f^*}^{\pm}, \delta_f^{\pm} \rangle = 1$ and $\langle \delta_{g^*}^{\pm}, \delta_g^{\pm} \rangle = 1$.

Definition 7.1.7. Let

$$\begin{aligned} \tilde{\omega}_{f^*} &:= G(\varepsilon_f^{-1})\omega_{\varepsilon_f}\omega_f \in M_{\mathrm{dR}}(f^*) \\ \tilde{\omega}_{g^*} &:= G(\varepsilon_g^{-1})\omega_{\varepsilon_g}\omega_g \in M_{\mathrm{dR}}(g^*). \end{aligned}$$

The next result is crucial for the period computation.

Lemma 7.1.8. For the Poincaré duality pairing $\langle \ , \ \rangle$ the following identities hold:

$$\begin{aligned} \langle \tilde{\omega}_{f^*}, \delta_f^{\pm} \rangle &= (2\pi i)^{-k-1}(-1)^k N_{\varepsilon_f} \langle \omega_f, \delta_{f^*}^{\mp} \rangle = \mp (2\pi i)^{-k-1}(-1)^k N_{\varepsilon_f} \langle \tilde{\omega}_{f^*}, \delta_{f^*}^{\mp} \rangle \\ \langle \tilde{\omega}_{g^*}, \delta_g^{\pm} \rangle &= (2\pi i)^{-k'-1}(-1)^{k'} N_{\varepsilon_g} \langle \omega_g, \delta_{g^*}^{\mp} \rangle = \mp (2\pi i)^{-k'-1}(-1)^{k'} N_{\varepsilon_g} \langle \tilde{\omega}_{g^*}, \delta_{g^*}^{\mp} \rangle. \end{aligned}$$

Proof. Using the definitions, one sees that

$$\begin{aligned} \langle \tilde{\omega}_{f^*}, \delta_f^{\pm} \rangle &= \langle G(\varepsilon_f^{-1})\omega_{\varepsilon_f}\omega_f, (2\pi i)^{-k-1}\delta_{\varepsilon_f}^{-1}\delta_{f^*}^{\mp} \rangle \\ &= (-1)^k N_{\varepsilon_f} \langle \delta_{\varepsilon_f}\omega_f, (2\pi i)^{-k-1}\delta_{\varepsilon_f}^{-1}\delta_{f^*}^{\mp} \rangle \\ &= (2\pi i)^{-k-1}(-1)^k N_{\varepsilon_f} \langle \omega_f, \delta_{f^*}^{\mp} \rangle. \end{aligned}$$

As $\overline{F_{\infty}^*}\omega_f = \overline{\omega}_{f^*}$ and $\overline{F_{\infty}^*}\delta_{f^*}^{\mp} = \mp \delta_{f^*}^{\mp}$ one gets also

$$\langle \tilde{\omega}_{f^*}, \delta_f^{\pm} \rangle = \mp (2\pi i)^{-k-1}(-1)^{k-1} N_{\varepsilon_f} \langle \overline{\omega}_{f^*}, \delta_{f^*}^{\mp} \rangle.$$

□

Finally, we define a basis for $M_B(f \otimes g)^*(-j-1)^+ \cong M_B(f^* \otimes g^*)(k+k'-j+1)^+$.

Definition 7.1.9. Let γ_j^*, δ_j^* be the L -basis of $M_B(f^* \otimes g^*)(k + k' - j + 1)^+$ defined by

$$\begin{aligned}\gamma_j^* &:= (2\pi i)^{-j-1} \delta_{f^*}^+ \otimes \delta_{g^*}^{(-1)^{j+1}} \\ \delta_j^* &:= (2\pi i)^{-j-1} \delta_{f^*}^- \otimes \delta_{g^*}^{(-1)^j}.\end{aligned}$$

We consider $\tilde{\omega}_{f^*} \otimes \tilde{\omega}_{g^*}$ as a basis of $\text{Fil}^{-j} M_{\text{dR}}(f \otimes g)^*$ via the isomorphism $M_{\text{dR}}(f \otimes g)^* \cong M_{\text{dR}}(f^* \otimes g^*)(k + k' + 2)$. With these definitions we are able to give an explicit generator of the L -vector space $\mathcal{R}(M(f \otimes g)^*(-j - 1))$.

Proposition 7.1.10. In the case where $j + 1$ is even (resp. odd) the image of the element

$$(2\pi i)^{-j-1} (\langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle)^{-1} \delta_j^* \quad (\text{resp. } (2\pi i)^{-j-1} (\langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^- \rangle)^{-1} \delta_j^*)$$

in $H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)\mathbf{R})$ is an L -basis of $\mathcal{R}(M(f \otimes g)^*(-j - 1))$.

Proof. We treat only the case of $j + 1$ even. The odd case is entirely similar. Write

$$\begin{aligned}\tilde{\omega}_{f^*} &= \langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \delta_{f^*}^+ + \langle \tilde{\omega}_{f^*}, \delta_f^- \rangle \delta_{f^*}^- \\ \tilde{\omega}_{g^*} &= \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle \delta_{g^*}^+ + \langle \tilde{\omega}_{g^*}, \delta_g^- \rangle \delta_{g^*}^-\end{aligned}$$

and let $\pi_{-j-1} : \mathbf{C} \rightarrow \mathbf{R}(-j - 1)$ be the projection $z \mapsto \frac{1}{2}(z + (-1)^{-j-1}z)$. Then one has (because $j + 1$ is even)

$$\begin{aligned}\pi_{-j-1}(\tilde{\omega}_{f^*} \otimes \tilde{\omega}_{g^*}) &= \langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle \delta_{f^*}^+ \otimes \delta_{g^*}^+ + \langle \tilde{\omega}_{f^*}, \delta_f^- \rangle \langle \tilde{\omega}_{g^*}, \delta_g^- \rangle \delta_{f^*}^- \otimes \delta_{g^*}^- \\ &= (2\pi i)^{j+1} (\langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle \gamma_j^* + \langle \tilde{\omega}_{f^*}, \delta_f^- \rangle \langle \tilde{\omega}_{g^*}, \delta_g^- \rangle \delta_j^*).\end{aligned}$$

The image of the element $(2\pi i)^{-j-1} (\langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle)^{-1} \delta_j^*$ in $H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)\mathbf{R})$ is a basis whose determinant with $\pi_{-j-1}(\tilde{\omega}_{f^*} \otimes \tilde{\omega}_{g^*})$ and γ_j^*, δ_j^* is 1. \square

With the above notations we get the formula for $L'(f, g, j + 1)$ as in Beilinson's conjecture. In the case $k = k' = 0$, i.e. $j = 0$, it was first proved by Beilinson [Bei84] and for general k, k' and $0 \leq j \leq \min\{k, k'\}$ it was announced by Scholl (unpublished, but see [Kin98] for closely related results in the Hilbert-Blumenthal case).

Theorem 7.1.11. Let $0 \leq j \leq \min\{k, k'\}$, then $\text{AJ}_{\mathcal{H}, f, g}(\text{Eis}_{\mathcal{H}, 1, N}^{[k, k', j]})$ generates the L -subspace

$$L'(f, g, j + 1) \mathcal{R}(M(f \otimes g)^*(-j - 1))$$

of $H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)\mathbf{R})$.

Proof. For simplicity we again treat only the case $j + 1$ even. From Theorem 6.2.9 we get

$$\begin{aligned}\left\langle \text{AJ}_{\mathcal{H}, f, g}(\text{Eis}_{\mathcal{H}, 1, N}^{[k, k', j]}), \frac{-1}{\langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} (\bar{\omega}_{f^*} \otimes \omega_g + (-1)^{j+1} \omega_f \otimes \bar{\omega}_{g^*}) \right\rangle = \\ \frac{(-1)^{k-j+1} (2\pi i)^{k+k'-2j}}{2 \langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} \frac{k!k'!}{(k-j)!(k'-j)!} L'(f, g, j + 1).\end{aligned}$$

and a straightforward computation with the basis $(2\pi i)^{-j-1} (\langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle)^{-1} \delta_j^*$ from Proposition 7.1.10 using Lemma 7.1.8 gives

$$\begin{aligned}\left\langle (2\pi i)^{-j-1} (\langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle)^{-1} \delta_j^*, \frac{-1}{\langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y} (\bar{\omega}_{f^*} \otimes \omega_g + (-1)^{j+1} \omega_f \otimes \bar{\omega}_{g^*}) \right\rangle = \\ \frac{(-1)^{k+k'} 2(2\pi i)^{k+k'-2j}}{N_{\varepsilon_f} N_{\varepsilon_g} \langle \omega_f, \bar{\omega}_f \rangle_Y \langle \omega_g, \bar{\omega}_g \rangle_Y}.\end{aligned}$$

This gives

$$\text{AJ}_{\mathcal{H}, f, g}(\text{Eis}_{\mathcal{H}, 1, N}^{[k, k', j]}) = L'(f, g, j + 1) \left(\frac{(-1)^{k'-j+1} N_{\varepsilon_f} N_{\varepsilon_g} k!k'!}{4(k-j)!(k'-j)!} \right) (2\pi i)^{-j-1} (\langle \tilde{\omega}_{f^*}, \delta_f^+ \rangle \langle \tilde{\omega}_{g^*}, \delta_g^+ \rangle)^{-1} \delta_j^*,$$

which implies the assertion of the theorem. \square

For the Perrin-Riou conjecture it is necessary to reformulate the above theorem in terms of a period.

Definition 7.1.12. Let $0 \leq j \leq \min\{k, k'\}$. The ∞ -period $\Omega_\infty(j+1)$ of the motive $M(f \otimes g)(j+1)$ is the element in $(L \otimes_{\mathbf{Q}} \mathbf{R})^\times$ given by the determinant of

$$0 \rightarrow \mathrm{Fil}^{-j} M_{\mathrm{dR}}(f \otimes g)_{\mathbf{R}}^* \rightarrow M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^+ \rightarrow H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}}) \rightarrow 0$$

with respect to the bases $\tilde{\omega}_{f^*} \otimes \tilde{\omega}_{g^*}$, γ_j^*, δ_j^* and $\mathrm{AJ}_{\mathcal{H}, f, g}(\mathrm{Eis}_{\mathcal{H}, b, N}^{[k, k', j]})$.

Remark 7.1.13. Note that $\Omega_\infty(j+1)$ is independent of the choice of bases up to an element in L^\times . The condition in the definition means that under the isomorphism

(7.1.3)

$$\det(\mathrm{Fil}^{-j} M_{\mathrm{dR}}(f \otimes g)_{\mathbf{R}}^*) \otimes \det(M_B(f \otimes g)^*(-j-1)_{\mathbf{R}}^+)^{-1} \otimes \det(H_{\mathcal{H}}^1(\mathbf{R}, M_B(f \otimes g)^*(-j)_{\mathbf{R}})) \cong L \otimes_{\mathbf{Q}} \mathbf{R}$$

the determinants of the corresponding bases map to $\Omega_\infty(j+1)$.

The next theorem is a reformulation of the Beilinson conjecture for the motive $M(f \otimes g)(j+1)$ with the ∞ -period $\Omega_\infty(j+1)$.

Theorem 7.1.14. Let $0 \leq j \leq \min\{k, k'\}$ and $L'(f, g, j+1) \in (L \otimes_{\mathbf{Q}} \mathbf{R})^\times$ be the leading term of the L -function of $M(f \otimes g)$ at $j+1$. Then

$$\frac{L'(f, g, j+1)}{\Omega_\infty(j+1)} = \frac{(-1)^{k'-j+1} 4 (k-j)! (k'-j)!}{N_{\varepsilon_f} N_{\varepsilon_g} k! k'!} \in L^\times.$$

Proof. This is just a reformulation of the computation in the proof of Theorem 7.1.11. \square

7.2. The Perrin-Riou conjecture (p-adic Beilinson conjecture). We continue to assume that f, g are new and $0 \leq j \leq \min\{k, k'\}$, and we choose a prime p not dividing the levels N_f, N_g , and a prime \mathfrak{P} above p of the coefficient field L , such that f and g are ordinary at \mathfrak{P} .

To formulate the Perrin-Riou conjecture we first have to define the p -adic period (see [PR95]). One would like to have a p -adic analogue of the complex period map

$$\alpha_{M(f \otimes g)(j+1)} : M_B(f \otimes g)(j+1)_{\mathbf{R}}^+ \rightarrow t(M(f \otimes g)(j+1))_{\mathbf{R}}.$$

The problem is that there is no good p -adic analogue of the two dimensional $+$ -part of Betti cohomology. To remedy this defect, Perrin-Riou (as explained by Colmez ([Col00]) proposes to choose elements $v_1, \dots, v_4 \in M_{\mathrm{dR}}(f \otimes g)_{\mathbf{Q}_p}$ such that $\mathcal{N}^+ := \langle v_1, v_2 \rangle$ plays the role of $M_B(f \otimes g)^+$ and $\mathcal{N}^- := \langle v_3, v_4 \rangle$ of $M_B(f \otimes g)^-$. The natural projection to the tangent space induces a map

$$\alpha_{M(f \otimes g)(j+1), \mathcal{N}} : \mathcal{N}^{(-1)^{j+1}} \rightarrow t(M(f \otimes g)(j+1))_{\mathbf{Q}_p}.$$

For suitable choices of \mathcal{N}^\pm one gets a short exact sequence

$$0 \rightarrow \ker(\alpha_{M(f \otimes g)(j+1), \mathcal{N}}) \rightarrow \mathcal{N}^{(-1)^{j+1}} \rightarrow t(M(f \otimes g)(j+1))_{\mathbf{Q}_p} \rightarrow 0$$

Using the pairing $M_{\mathrm{dR}}(f \otimes g)(j+1) \times M_{\mathrm{dR}}(f \otimes g)^*(-j) \rightarrow L$ we can identify $t(M(f \otimes g)^*(-j)) \cong (\mathrm{Fil}^{j+1} M_{\mathrm{dR}}(f \otimes g))^*$ and get

$$(7.2.1) \quad 0 \rightarrow \mathrm{Fil}^{-j} M_{\mathrm{dR}}(f \otimes g)_{\mathbf{Q}_p}^* \rightarrow (\mathcal{N}^{(-1)^{j+1}})^* \rightarrow \ker(\alpha_{M(f \otimes g)(j+1), \mathcal{N}})^* \rightarrow 0.$$

If we compose the Abel-Jacobi map as in §5.4 with the canonical projection we get

$$(7.2.2) \quad H_{\mathrm{syn}}^3(Y_1(N)_{\mathbf{Z}_p}^2, \mathrm{TSym}^{[k, k']}(\mathcal{H}_{\mathbf{Q}_p})(2-j)) \longrightarrow t(M(f \otimes g)^*(-j))_{\mathbf{Q}_p} \longrightarrow \ker(\alpha_{M(f \otimes g)(j+1), \mathcal{N}})^*.$$

In our case we will choose $\mathcal{N}^+ = \mathcal{N}^-$:

Definition 7.2.1. Assume that $\alpha_f, \alpha_g \in L$ and let η_f^α, η_g' and ω_g' be the classes defined in 6.5.1, 6.1.3 and 6.1.1 respectively, then we put

$$v_1 = v_3 = \frac{N_{\varepsilon_f} N_{\varepsilon_g}}{\mathcal{E}(f) \mathcal{E}^*(f)} \eta_f^\alpha \otimes \omega_g' \quad \text{and} \quad v_2 = v_4 = \eta_f^\alpha \otimes \eta_g'.$$

so that $\mathcal{N} := \mathcal{N}^+ = \mathcal{N}^- = \eta_f^\alpha L_{\mathfrak{P}} \otimes M_{\mathrm{dR}}(g)_{\mathbf{Q}_p} \subset M_{\mathrm{dR}}(f \otimes g)_{\mathbf{Q}_p}$.

Remark 7.2.2. Perrin-Riou associates to each choice of v_1, v_2, v_3, v_4 a p -adic L -function. In this paper we will restrict ourselves to a choice which gives the p -adic L -function $L_p(f, g, s)$. The above formulae only define v_2 modulo v_1 , but this is not important for the constructions below.

The p -adic period is now defined as follows:

Definition 7.2.3. The p -adic period $\Omega_p(j+1, \underline{v})$ associated to $\underline{v} := (v_1, v_2, v_3, v_4)$ is defined to be

$$\Omega_p(j+1, v) := \left\langle \text{AJ}_{\text{syn}, f, g} \left(\text{Eis}_{\text{syn}, 1, N}^{[k, k', j]} \right), v_1 \right\rangle$$

where one considers $\text{AJ}_{\text{syn}, f, g} \left(\text{Eis}_{\text{syn}, 1, N}^{[k, k', j]} \right)$ as an element in $\ker(\alpha_{M(f \otimes g)(j+1), \mathcal{N}})^*$ using 7.2.2.

Remark 7.2.4.

- (i) Colmez chooses a splitting of the exact sequence 7.2.1 and shows that his definition does not depend on this choice. We have taken $v_2 = v_4 = \eta_f^\alpha \otimes \varpi_{g^*}$ to be the element mapping to a generator of $t(M(f \otimes g)(j+1))_{\mathbf{Q}_p}$.
- (ii) From the definition and Theorem 6.5.9 one sees that $\Omega_p(j+1, v) \neq 0$ precisely if the p -adic L -function $L_p(f, g, s)$ does not vanish at $s = j+1$. We point out again, that for our choice of j , this is not a point of classical interpolation so that we have no control on this vanishing in terms of the complex L -function.

For $x \in M_{\text{dR}}(f \otimes g)_{\mathbf{Q}_p}$ write, as in Colmez [Col00], $t^{-n}x \in M_{\text{dR}}(f \otimes g)(n)$ for the image of x under the canonical isomorphism. Note that this is compatible with the isomorphisms $M_{\text{dR}}(f \otimes g)(n)_{\mathbf{Q}_p} \cong D_{\text{cris}}(M(f \otimes g)(n)) = t^{-n}D_{\text{cris}}(M(f \otimes g)) \cong M_{\text{dR}}(f \otimes g)_{\mathbf{Q}_p}$ and that φ acts via p^{-n} on t^{-n} . Then an easy computation using $\varphi(\eta_f^\alpha) = \alpha_f \eta_f^\alpha$ gives

$$\begin{aligned} \det(1 - \varphi | t^{-j-1}\mathcal{N}) &= \left(1 - \frac{\alpha_f \alpha_g}{p^{j+1}}\right) \left(1 - \frac{\alpha_f \beta_g}{p^{j+1}}\right) \\ \det(1 - p^{-1}\varphi^{-1} | t^{-j-1}\mathcal{N}) &= \left(1 - \frac{p^j}{\alpha_f \alpha_g}\right) \left(1 - \frac{p^j}{\alpha_f \beta_g}\right) \end{aligned}$$

so that

$$(7.2.3) \quad \mathcal{E}(f, g, j+1) = \frac{\det(1 - p^{-1}\varphi^{-1} | t^{-j-1}\mathcal{N}) \det(1 - \varphi | D_{\text{cris}}(M(f \otimes g)(j+1)))}{\det(1 - \varphi | t^{-j-1}\mathcal{N})}.$$

Definition 7.2.5. Denote by

$$L'_{\{p\}}(f, g, j+1)$$

the derivative of $L(f, g, s)$ at $s = j+1$ without the Euler factor at p , which is $\det(1 - \varphi | D_{\text{cris}}(M(f \otimes g)(j+1)))$.

The next Theorem is Perrin-Riou's conjecture [PR95, 4.2.2] (see also Colmez [Col00, Conjecture 2.7]) for Hida's p -adic Rankin-Selberg L -function.

Theorem 7.2.6. Let $L_p(f, g, s)$ be Hida's p -adic Rankin-Selberg L -function and let $0 \leq j \leq \min(k, k')$. Suppose that $\Omega_p(j+1, v) \neq 0$ holds. Then

$$L_p(f, g, j+1) = 4^{-1}(-1)^{k'+1}\Gamma(j+1)\Gamma(j-k')^* G(\varepsilon_f^{-1})G(\varepsilon_g^{-1}) \frac{L'_{\{p\}}(f, g, j+1)}{\Omega_\infty(j+1)} \Omega_p(j+1, \frac{1-p^{-1}\varphi^{-1}}{1-\varphi}(-t)^{-j-1}\underline{v}),$$

where $\Gamma(j-k')^* = \frac{(-1)^{k'-j}}{(k'-j)!}$ is the residue of $\Gamma(s)$ at $s = j-k'$.

Proof. By Theorem 6.5.9 and the above calculation of the action of φ one has

$$\begin{aligned} \Omega_p(j+1, \frac{1-p^{-1}\varphi^{-1}}{1-\varphi}(-t)^{-j-1}\underline{v}) &= (-1)^{j+1} \frac{\det(1 - p^{-1}\varphi^{-1} | t^{-j-1}\mathcal{N})}{\det(1 - \varphi | t^{-j-1}\mathcal{N})} \left\langle \text{AJ}_{\text{syn}, f, g} \left(\text{Eis}_{\text{syn}, 1}^{[k, k', j]} \right), v_1 \right\rangle \\ &= (-1)^{k'}(k')! \binom{k}{j} \frac{G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})N_{\varepsilon_f}N_{\varepsilon_g}}{\mathcal{E}(f, g, j+1)} \frac{\det(1 - p^{-1}\varphi^{-1} | t^{-j-1}\mathcal{N})}{\det(1 - \varphi | t^{-j-1}\mathcal{N})} L_p(f, g, j+1), \end{aligned}$$

which gives using (7.2.3)

$$\Omega_p(j+1, \frac{1-p^{-1}\varphi^{-1}}{1-\varphi}(-t)^{-j-1}\underline{v}) = (-1)^{k'}(k')! \binom{k}{j} \frac{G(\varepsilon_f^{-1})G(\varepsilon_g^{-1})N_{\varepsilon_f}N_{\varepsilon_g}}{\det(1 - \varphi | D_{\text{cris}}(M(f \otimes g)(j+1)))} L_p(f, g, j+1).$$

On the other hand, by Theorem 7.1.14 we have

$$4^{-1}\Gamma(j+1)\Gamma(j-k')^* \frac{L'_{\{p\}}(f, g, j+1)}{\Omega_\infty(j+1)} = \frac{-1}{N_{\varepsilon_f}N_{\varepsilon_g}k'!} \binom{k}{j}^{-1} \det(1 - \varphi | D_{\text{cris}}(M(f \otimes g)(j+1))),$$

which proves the theorem. \square

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